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Essential points of the n-cube subset partitioning characterisation

Hasmik Sahakyan *

Institute for Informatics and Automation Problems, National Academy of Sciences, 1 P. Sevak st., 0014 Yerevan, Armenia

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ABSTRACT

The question of necessary and sufficient conditions for the existence of a simple hypergraph with a given degree sequence is a long-standing open problem. Let $\psi_m(n)$ denote the set of all degree sequences of simple hypergraphs on vertex set $[n] = \{1, 2, \cdots, n\}$ that have m edges. A simple characterisation of $\psi_m(n)$ is defined in terms of its upper and/or lower elements (degree sequences). In the process of finding all upper degree sequences, a number of results have been achieved in this paper. Classes of upper degree sequences with lowest rank (sum of degrees) $r_{\min}$ and with highest rank $r_{\max}$ have been found; in the case of $r_{\min}$, the unique class of isomorphic hypergraphs has been determined; the case of $r_{\max}$ leads to the simple uniform hypergraph degree sequence problem. A smaller generating set has been found for $\psi_m(n)$. New classes of upper degree sequences have been generated from the known sequences in dimensions less than n.

1. Introduction

Let E^n be the set of vertices of the n-dimensional unit cube, $E^n = \{(x_1, \ldots, x_n)/x_i \in \{0, 1\}, i = 1, \ldots, n\}$. For an arbitrary variable x_i consider the partition/splitting of E^n into two $(n-1)$-dimensional subcubes of E^n according to the value of x_i:

$$E^{n-1}_{x_i=1} = \{(x_1, \ldots, x_n) \in E^n/x_i = 1\}, \qquad E^{n-1}_{x_i=0} = \{(x_1, \ldots, x_n) \in E^n/x_i = 0\}.$$

Each set $\mathcal{E} \subseteq E^n$ will have (empty or non-empty) subsets in these subcubes: $\mathcal{E}_{x_1=1}$ and $\mathcal{E}_{x_1=0}$. Similarly, E^n can be split according to more than one variable. Splitting by the set of variables $x_{i_1}, \ldots, x_{i_k}$ we obtain 2^k $(n-k)$-dimensional subcubes, where the values of the variables $x_{i_1}, \ldots, x_{i_k}$ are appropriately fixed in each of them. The notations are defined in a similar manner. For example, $E^{n-k}_{x_{i_1}=1,\ldots,x_{i_k}=1} = \{(x_1, \ldots, x_n) \in E^n/x_{i_1} = 1, \ldots, x_{i_k} = 1\}$, and $\mathcal{E}_{x_{i_1}=1,\ldots,x_{i_k}=1}$ denotes the part of $\mathcal{E}$ in $E^{n-k}_{x_{i_1}=1,\ldots,x_{i_k}=1}$.

An integer vector $d = (d_1, \ldots, d_n)$ is called the *associated vector of partitions* (1-partitions) of the set $\mathcal{E} \subseteq E^n$ if $d_i = |\mathcal{E}_{x_i=1}|$ for all i, $1 \le i \le n$. In general, different sets may have the same associated vector of partitions. We consider the question of the existence of vertex sets in E^n with the given associated vector of partitions. This problem is known also in terms of families of sets or hypergraphs.

Consider the power set of $[n] = \{1, 2, \ldots, n\}$ and its partial order by inclusion. Identify subsets of $[n]$ with (0, 1)-sequences of length n such that the i-th entry equals '1' if and only if the i-th element of $[n]$ is included in the subset. In this manner, we obtain a mapping of the power set into E^n. Each set $\mathcal{E} \subseteq E^n$ can be identified with a family of subsets of $[n]$ or with a simple hypergraph H on vertex set $[n]$, whose edges are determined by the elements of $\mathcal{E}$. The degree of a vertex i is equal to $|\mathcal{E}_{x_i=1}|$. In these terms the existence of vertex sets in E^n with the given associated vector of partitions becomes

* Tel.: +374 55 511231; fax: +374 10 284435.
E-mail address: hasmik@ipia.sci.am.

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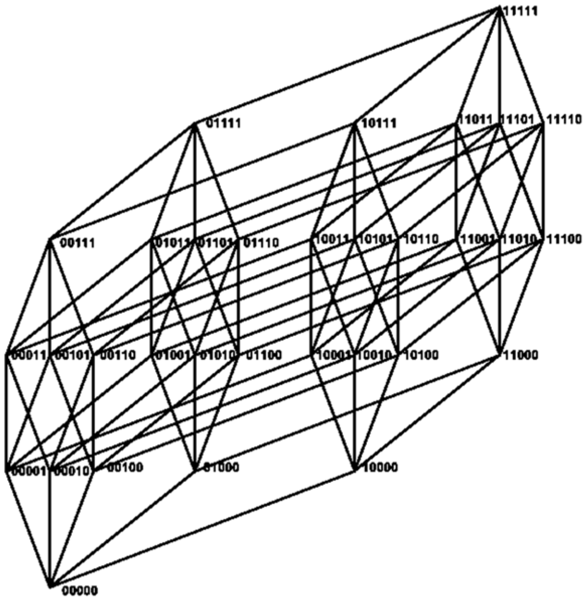

Fig. 1. Hasse diagram of the 5-dimensional unit cube.

equivalent to the existence of a simple hypergraph H with the given degree sequence. This is a long-standing open problem known as the hypergraph degree sequence problem [1]. The hypergraph degree sequence problem has been investigated by several authors [8,2,3,6,4]. In particular, the polytope of degree sequences of r-uniform hypergraphs was studied in [2] and some partial information was obtained. It was proved in [6] that any two 3-uniform hypergraphs with the same degree sequence can be transformed into each other using a sequence of trades. Steepest degree sequences were defined in [3] and it was proved that the entire set of degree sequences of simple uniform hypergraphs can be determined by its steepest elements.

For a given m, $0 \le m \le 2^n$, let $\psi_m(n)$ denote the set of all degree sequences of simple hypergraphs on the vertex set $[n]$ that have m edges. The upper and lower elements of $\psi_m(n)$ were defined in [8], where it was proved that $\psi_m(n)$ can be determined by the set of its upper and/or lower elements. In this paper, we present results which may be useful in finding all upper degree sequences. In Section 2, we find classes of upper degree sequences with lowest rank (sum of degrees) $r_{\min}$ and with highest rank $r_{\max}$. We determine $\mathcal{H}_{rmin}$ and $\mathcal{H}_{rmax}$, the corresponding classes of simple hypergraphs. While $\mathcal{H}_{rmin}$ is unique up to isomorphism class, the characterisation of $\mathcal{H}_{rmax}$ leads to the hypergraph degree sequence problem for simple uniform hypergraphs. Section 3 is devoted to finding a smaller generating set for $\psi_m(n)$; we prove that $\psi_m(n)$ can be determined by the intersection of the set of its upper elements and the set of its steepest elements. Section 4 deals with the process of generating new classes of upper degree sequences from the known sequences of dimension less than n.

We present the results in terms of hypergraphs and degree sequences. However, where it is technically reasonable, we use the means of E^n and a splitting of E^n in one or two directions. We use the geometrical mapping of E^n by the Hasse diagram. The diagram has $n+1$ levels numbered from 0 (the lowest level) to n; the k-th level contains all vertices that have k entries equal to 1. Edges connect those vertices in neighbouring levels that are related by a cover relation. Fig. 1 shows the Hasse diagram of E^5.

2. Generating $\psi_m(n)$ using the upper elements

2.1. Preliminaries

For a given m, $0 \le m \le 2^n$, let $\psi_m(n)$ denote the set of all degree sequences of simple hypergraphs on the vertex set $[n]$ that have m edges. Define the greed Ξ^n_{m+1} as $\Xi^n_{m+1} = \{(a_1, \ldots, a_n) : 0 \le a_i \le m \text{ for all } i\}$ and place a component-wise partial order on Ξ^n_{m+1}: $(a_1, \ldots, a_n) \le (b_1, \ldots, b_n)$ if and only if $a_i \le b_i$ for all i. This order makes Ξ^n_{m+1} a ranked partially ordered set (poset) for which the rank of an element $(a_1, \ldots, a_n)$ is given by $a_1 + \cdots + a_n$. In this manner, $\psi_m(n)$ is a subset of Ξ^n_{m+1}.

Definition 1 (*[8]*). A degree sequence $(d_1, \ldots, d_n) \in \psi_m(n)$ is called an upper (lower) degree sequence for $\psi_m(n)$ if no $(a_1, \ldots, a_n) \in \Xi^n_{m+1}$ with $(a_1, \ldots, a_n) > (d_1, \ldots, d_n)$ $((a_1, \ldots, a_n) < (d_1, \ldots, d_n))$ belongs to $\psi_m(n)$.

Denote by $\hat{\psi}_m(n)$ and $\check{\psi}_m(n)$ the sets of all upper and lower degree sequences of $\psi_m(n)$ respectively.

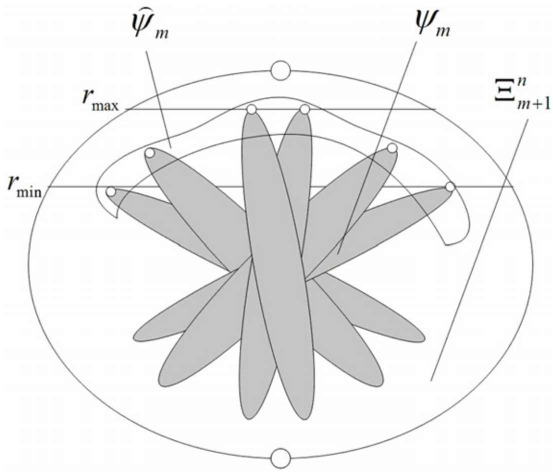

Fig. 2. Illustration of upper degree sequences.

List of some properties of upper/lower degree sequences from [8]:

($P1$). $\psi_m(n)$ is symmetric, and elements of $\hat{\psi}_m(n)$ and $\check{\psi}_m(n)$ appear in pairs: for each $\hat{d} \in \hat{\psi}_m(n)$, there exists a $\check{d}$ in $\check{\psi}_m(n)$ that is obtained from $\hat{d}$ by inverting its coordinates, $\check{d}_i = m - \hat{d}_i$, and vice versa. We call such elements opposites. Hence $|\hat{\psi}_m(n)| = |\check{\psi}_m(n)|$.

($P2$). $\hat{d}_i \geq m/2$ for each $\hat{d} \in \hat{\psi}_m(n)$, and $\check{d}_i \leq m/2$ for each $\check{d} \in \check{\psi}_m(n)$.

($P3$). $\hat{\psi}_m$ and $\check{\psi}_m$ are antichains in Ξ^n_{m+1}.

Consider the following special subposets in Ξ^n_{m+1}: $I(\check{d}, \hat{d}) = \{a \in \Xi^n_{m+1} / \check{d} \leq a \leq \hat{d}\}$, where $(\check{d}, \hat{d})$ is a pair of opposite elements from $\check{\psi}_m(n)$ and $\hat{\psi}_m(n)$.

Theorem 1 (*[8]*). $\psi_m(n) = \bigcup_{\hat{d} \in \hat{\psi}_m(n)} I(\check{d}, \hat{d})$.

Thus, $\hat{\psi}_m(n)$ and/or $\check{\psi}_m(n)$ can be considered as generating sets for constructing all elements of $\psi_m(n)$. We restrict our attention to $\hat{\psi}_m(n)$. It is worth noting the relation of $\hat{\psi}_m(n)$ to the monotone Boolean functions defined on E^n. Each set of vertices of E^n can be identified with the set of 1 values of some Boolean function. In this manner, monotone Boolean functions represent a specific class of sets in E^n. Consider M^1_m, the class of m-sets in E^n represented by monotone Boolean functions with m values 1, and let $\psi^{M_1}_m(n)$ denote the class of corresponding associated vectors of partitions/degree sequences. It was proved in [8] that $\hat{\psi}_m(n) \subset \psi^{M_1}_m(n)$. Thus, one method for determining all elements of $\hat{\psi}_m(n)$ is to construct all monotone Boolean functions that have m values 1.

2.2. Upper elements with lowest rank

Consider the Hasse diagram of Ξ^n_{m+1}. It has $m \cdot n + 1$ levels according to the ranks of elements: the i-th level contains all elements at rank i. Suppose that $r_{\min}$ and $r_{\max}$ are the lowest and highest ranks for upper degree sequences. Obviously $r_{\max}$ is the highest rank for any degree sequence. An illustration is given in Fig. 2.

In this subsection we define a class $\mathcal{H}_{rmin}$ and prove that $\mathcal{H}_{rmin}$ is the unique up to isomorphism class of simple hypergraphs that have upper degree sequences with lowest rank.

Consider the reverse lexicographic ordering on E^n: $(\alpha_1, \ldots, \alpha_n) < (\beta_1, \ldots, \beta_n)$ if and only if the numeric value of $(\alpha_1, \ldots, \alpha_n)$ is greater than that of $(\beta_1, \ldots, \beta_n)$. Observe that the first 2^{n-1} elements compose $E^{n-1}_{x_1=1}$ and the remaining 2^{n-1} elements compose $E^{n-1}_{x_1=0}$; both are in the reverse lexicographic order of elements. Continue recursively with this observation: the first 2^t elements, and consequently, each 2^t elements for arbitrary t, compose t-dimensional subcubes. The values of $x_1, \ldots, x_{n-t}$ are appropriately fixed in each of them. Let $R_m(n)$ denote the initial m-length sequence of the reverse lexicographic ordering on E^n. It is easy to check that $R_m(n)$ corresponds to the set of 1 values of a monotone Boolean function. The structure of this set can be illustrated as in Fig. 3, where $k_1, \ldots, k_p$ are parameters in binary representation of $m : m = 2^{k_1} + \cdots + 2^{k_p}$.

($P4$). Let $2^t < m \leq 2^{t+1}$. The initial m-sequence of the reverse lexicographic ordering on E^n corresponds to that on $E^t_{x_1=1,\ldots,x_{n-t-1}=1}$; it consists of the entire t-dimensional subcube $E^t_{x_1=1,\ldots,x_{n-t}=1}$, and the remaining elements compose the initial $(m - 2^t)$-sequence in $E^t_{x_1=1,\ldots,x_{n-t}=0}$.

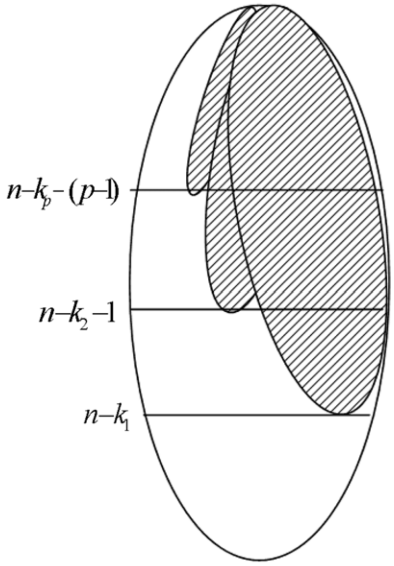

Fig. 3. The structure of the initial segment of the reverse lexicographic ordering.

$(P5)$. Let $(d_1, \dots, d_n)$ denote the associated vector of partitions of $R_m(n)$. Then, d_1 is the largest possible value of the partition size for any m-set, d_2 is the next largest value for fixed d_1, and so forth. Obviously, $d_1 \geq \cdots \geq d_n$.

If the lexicographic ordering applies to all permutations of $(x_1, \dots, x_n)$, then the initial m-sequences in all these orderings will induce a class of isomorphic monotone Boolean functions. Let $\mathcal{H}_{rmin}$ denote the corresponding class of isomorphic hypergraphs and $d(\mathcal{H}_{rmin})$ denote the class of their degree sequences.

Before formulation of the next theorem we present the following lemma that helps to narrow our attention to the case $m \leq 2^{n-1}$.

Lemma 1. *Let $\mathcal{E}_1 \in M_m^1$, and let $(d_1, \dots, d_n)$ be the association vector of partitions of $\mathcal{E}_1$. Then, there exists $\mathcal{E}_2 \in M_{2^n-m}^1$ with the associated vector of partitions equal to* $(2^{n-1} - m + d_1, \dots, 2^{n-1} - m + d_n)$.

Proof. The lemma has been proved by taking the complement of $\mathcal{E}_1$ in E^n, and then inverting it. □

According to Lemma 1 we consider $1 < m \leq 2^{n-1}$ omitting $m = 1$ as obvious. Suppose that $2^k < m \leq 2^{k+1}$ for some k, and thus $0 \leq k \leq n-2$.

Theorem 2. $d(\mathcal{H}_{rmin}) \subseteq \hat{\psi}_m(n)$.

Proof. Let $\mathcal{H} \in \mathcal{H}_{rmin}$ and $(d_1, \dots, d_n)$ is the degree sequence of $\mathcal{H}$. Assume that $\mathcal{E}$ is the m-set in E^n, that represents the edges of $\mathcal{H}$. We perform the proof by induction on n. It is easy to confirm that the assertion is true for small n. We assume it is true for all dimensions $\leq n-1$ and prove it for n. Suppose for a contradiction that there exists $(s_1, \dots, s_n) \in \hat{\psi}_m(n)$, with $(s_1, \dots, s_n) > (d_1, \dots, d_n)$. Let $\mathcal{A}$ be an m-set in E^n, with partition sizes given by $(s_1, \dots, s_n)$. By Property $P4$, $\mathcal{E}$ is the initial ($\leq 2^{k+1}$)-sequence of the reverse lexicographic ordering on $E^{k+1}_{x_1=1,\dots,x_{n-k-1}=1}$, which implies that $d_1 = \cdots = d_{n-k-1} = m$. Then, $s_1 = \cdots = s_{n-k-1} = m$; hence, $\mathcal{A}$ is included in $E^{k+1}_{x_1=1,\dots,x_{n-k-1}=1}$ as well. This moves consideration of $\mathcal{E}$ and $\mathcal{A}$ into the $(k+1)$-dimensional unit cube E^{k+1}. Now the corresponding degree sequences are as follows: $(d_{n-k}, \dots, d_n)$ and $(s_{n-k}, \dots, s_n)$, and $(s_{n-k}, \dots, s_n) > (d_{n-k}, \dots, d_n)$. This contradicts the induction hypothesis. □

Theorem 3. *A degree sequence of $\hat{\psi}_m(n)$ has rank $r_{\min}$ if and only if it belongs to $d(H_{rmin})$.*

Proof. The initial sequence of lexicographic ordering of E^n is known as a unique (up to isomorphism) solution for a number of well known problems, such as edge and vertex isoperimetry, optimal assignment of numbers to vertices, and maximum/minimum weight ideal in the partial order of E^n (see e.g., [5]). In terms of degree sequences the result can be expressed as: a degree sequence of $\psi_m^{M_1}(n)$ has rank $r_{\min}$ if and only if it belongs to $d(H_{rmin})$. On the other hand, by Theorem 2, $d(\mathcal{H}_{rmin}) \subseteq \hat{\psi}_m(n)$, which completes the proof. □

The structure of hypergraphs of $\mathcal{H}_{rmin}$, given in Fig. 3, helps to easily calculate $r_{\min}$, as well as components of degree sequences.

$$r_{\min} = \sum_{i=1}^{n} d_i = \sum_{i=1}^{p} \left(\left(n - k_i - (i-1) \right) \cdot 2^{k_i} + k_i \cdot 2^{k_i - 1} \right).$$

$$d_i = \left(\sum_{l=1}^{j-1} 2^{k_l-1}\right) + 2^{k_j}, \quad \text{for } i = k_j+1, j = 1, \dots, p,$$

$$d_i = \left(\sum_{l=1}^{j} 2^{k_l-1}\right) + \left(\sum_{l=j+1}^{p} 2^{k_l}\right), \quad \text{for } k_{j+1}+2 \le i \le k_j, j = 1, \dots, p-1,$$

$$d_i = \sum_{l=1}^{p} 2^{k_l-1} = m/2, \quad \text{for } 1 \le i \le k_p,$$

$$d_i = \sum_{l=1}^{p} 2^{k_l} = m, \quad \text{for } k_1+2 \le i \le n.$$

2.3. Upper elements with highest rank

Let m is represented in canonical form: $m = C_n^n + C_n^{n-1} + \dots + C_n^{n-k} + \delta, \delta \le C_n^{n-k-1}$.

Define a specific class of hypergraphs on the vertex set $[n]$. Take all vertices of n-th, $(n-1)$-th, ..., $(n-k)$-th levels and δ vertices from $(n-k-1)$-th level of E^n—as edges of hypergraphs. The choice of δ vertices in level $(n-k-1)$ is arbitrary. Obviously, we obtain $\mathcal{H}_{rmax}$, the class of all hypergraphs with the highest rank $r_{\max}$, and $r_{\max} = \sum_{i=0}^{k}(n-i)\cdot C_n^{n-i} + (n-k-1)\cdot\delta$.

For each $\mathcal{H} \in \mathcal{H}_{rmax}$, each degree d_i can be represented as: $d_i = \sum_{j=0}^{k} C_{n-1}^{n-j-1} + s_i$, where the first summand is constant, and s_i comes from δ elements. Separate the sequence $(s_1, \dots, s_n)$. This represents the degree sequence of $(n-k-1)$-uniform hypergraph on $[n]$ with edges determined by these δ vertices of E^n.

Thus, the question of characterisation of degree sequences of $\mathcal{H}_{rmax}$ is equivalent to the characterisation of degree sequences of simple uniform hypergraphs. This problem is open even for 3-uniform hypergraphs [2,3,6].

Before formulation of the next theorem we present a lemma for which an analogue in terms of the unit cube is proved in [7].

Lemma 2. *Let $m = C_n^n + C_n^{n-1} + \dots + C_n^{n-k} + \delta$. There exists a hypergraph in $\mathcal{H}_{rmax}$ such that for all i either $d_i = \sum_{j=0}^{k} C_{n-1}^{n-j-1} + \lfloor \frac{(n-k-1)\cdot\delta}{n} \rfloor$ or $d_i = \sum_{j=0}^{k} C_{n-1}^{n-j-1} + \lfloor \frac{(n-k-1)\cdot\delta}{n} \rfloor + 1$.*

Lemma 2 implies that $\sum_{j=0}^{k} C_{n-1}^{n-j-1} + \lfloor \frac{(n-k-1)\cdot\delta}{n} \rfloor$ is the greatest degree that can have regular hypergraphs. Applying now Theorem 1 we obtain the following necessary and sufficient condition for the regular hypergraph degree sequences.

Theorem 4. *There exists simple s-regular hypergraph on $[n]$ with m edges if and only if $m - \left(\sum_{j=0}^{k} C_{n-1}^{n-j-1} + \lfloor \frac{(n-k-1)\cdot\delta}{n} \rfloor\right) \le s \le \sum_{j=0}^{k} C_{n-1}^{n-j-1} + \lfloor \frac{(n-k-1)\cdot\delta}{n} \rfloor$.*

3. Generating $\psi_m(n)$ using the upper steepest elements

In this section we introduce another resource from [3], steepest degree sequences for constructing all elements of $\psi_m(n)$.

Definition 2 (*[3]*). Let d and d' be finite decreasing sequences of nonnegative integers. d' is an elementary flattening of d if and only if d' can be obtained from d by

1. finding i, j such that $d_i \ge d_j + 2$ and then,
2. transferring 1 from d_i to d_j, $d'_i = d_i - 1$ and $d'_j = d_j + 1$, and
3. re-ordering the resulting sequence such that it is decreasing.

Definition 3 (*[3]*). Let d and d' be finite decreasing sequences of nonnegative integers. d' is flatter than d and d is steeper than d' if and only if d' can be obtained from d by a non-empty sequence of elementary flattenings.

Now, we formulate a theorem for which an analogue for uniform hypergraphs is proved in [3].

Theorem 5. *If d belongs to $\psi_m(n)$, then all d' flatter than d also belong to $\psi_m(n)$.*

Proof. Let $d = (d_1, \dots, d_n) \in \psi_m(n)$ and $\mathcal{E}$ be a set in E^n for which d is the associated vector of partitions. Suppose that d' is flatter than d. Without loss of generality, we can assume that d' is an elementary flattening of d. Then, there exist i, j such that $d_i \ge d_j + 2$ and $d' = (d_1, \dots, d_i - 1, \dots, d_j + 1, \dots, d_n)$. Now, split E^n according to two variables x_i and x_j: $\mathcal{E}^{n-2}_{x_i=1,x_j=1}$, $\mathcal{E}^{n-2}_{x_i=1,x_j=0}$, $\mathcal{E}^{n-2}_{x_i=0,x_j=1}$, and $\mathcal{E}^{n-2}_{x_i=0,x_j=0}$ denote the parts of $\mathcal{E}$ in $E^{n-2}_{x_i=1,x_j=1}$, $E^{n-2}_{x_i=1,x_j=0}$, $E^{n-2}_{x_i=0,x_j=1}$, and $E^{n-2}_{x_i=0,x_j=0}$, respectively. Then, $|\mathcal{E}^{n-2}_{x_i=1,x_j=1}| + |\mathcal{E}^{n-2}_{x_i=1,x_j=0}| = d_i$ and $|\mathcal{E}^{n-2}_{x_i=1,x_j=1}| + |\mathcal{E}^{n-2}_{x_i=0,x_j=1}| = d_j$; hence, $|\mathcal{E}^{n-2}_{x_i=1,x_j=0}| - |\mathcal{E}^{n-2}_{x_i=0,x_j=1}| \ge 2$. Thus, we can move one vertex from $\mathcal{E}^{n-2}_{x_i=1,x_j=0}$ to $\mathcal{E}^{n-2}_{x_i=0,x_j=1}$. This operation will provide the necessary $d_i - 1$ and $d_j + 1$ values. A geometrical visualisation is provided in Fig. 4. □

Definition 4 (*[3]*). $d \in \psi_m(n)$ is a steepest degree sequence for $\psi_m(n)$ if and only if all d' steeper than d do not belong to $\psi_m(n)$.

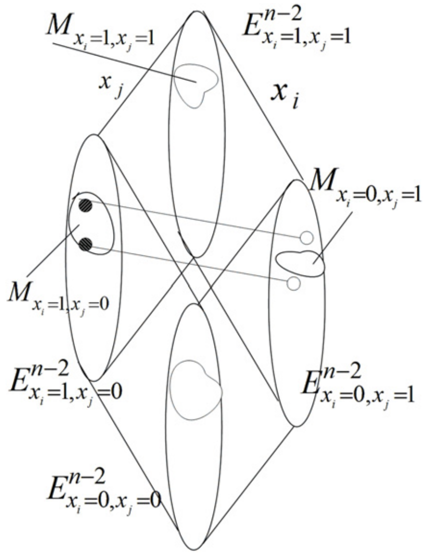

Fig. 4. Splitting of the cube and replacement of vertices.

It follows from Theorem 5 that the steepest sequences of $\psi_m(n)$ on each level of Ξ^n_{m+1} determine all sequences of $\psi_m(n)$ in that level. Thus, the process of generating the sequences of $\psi_m(n)$ using the steepest elements proceeds level by level of Ξ^n_{m+1}, whereas the generation of $\psi_m(n)$ using the upper elements proceeds through subposets of Ξ^n_{m+1}. Even for 3-uniform hypergraphs, the number of steepest degree sequences is exponential [3]. The number of upper sequences is unknown, but if we retrieve these sequences from $\psi^{M_1}_m(n)$, then we must consider all monotone Boolean functions that have m values 1. We prove by Theorem 6 that $\psi_m(n)$ can be determined by the intersection of two sets: the set of its upper elements and the set of its steepest elements. This result decreases the size of the generating set for $\psi_m(n)$.

First, we observe that the set of all steepest degree sequences of $\psi_m(n)$ can contain both upper sequences and non-upper sequences. Similarly, $\hat{\psi}_m(n)$ contains both steepest and non-steepest sequences. Let $\hat{\psi}^S_m(n)$ denote the set of upper steepest elements of $\psi_m(n)$. We illustrate these sets for the following example: $n = 5$ and $m = 14$. One method of composing $\hat{\psi}_{14}(5)$ is to construct all monotone Boolean functions in E^5 with 14 values equal to 1, compose $\psi^{M^1}_{14}(5)$, and then remove all non-upper sequences. We devised and programmed a recursive algorithm for generating monotone Boolean functions and composed $\psi^{M^1}_{14}(5)$ (only the decreasing sequences of $\psi^{M^1}_{14}(5)$ are presented):

$$\begin{aligned}\psi^{M^1}_{14}(5) = \{&(14,8,8,8,7),(13,9,9,8,8),(12,10,10,8,7),(12,10,9,9,8),(12,10,9,8,8),\\&(12,9,9,9,9),(11,11,10,8,8),(11,11,9,9,8),(11,10,10,9,9),\\&(11,10,10,9,8),(11,10,9,9,9),(10,10,10,10,9),(10,10,10,10,7)\}.\end{aligned}$$

$$\begin{aligned}\hat{\psi}_{14}(5) = \{&(14,8,8,8,7),(13,9,9,8,8),(12,10,10,8,7),(12,10,9,9,8),(12,9,9,9,9),\\&(11,11,10,8,8),(11,11,9,9,8),(11,10,10,9,9),(10,10,10,10,9)\}.\end{aligned}$$

$$\hat{\psi}^S_{14}(5) = \{(14,8,8,8,7),(13,9,9,8,8),(12,10,10,8,7),(12,10,9,9,8),(11,11,10,8,8),(11,10,10,9,9)\}.$$

Theorem 6. *If d is an upper degree sequence of $\psi_m(n)$, then all d' from $\psi_m(n)$ that are steeper than d are also upper degree sequences.*

Proof. Let $d = (d_1, \ldots, d_n)$ be an upper degree sequence of $\psi_m(n)$. Without loss of generality, we can assume that d is a decreasing sequence. Assume that $d' = (d'_1, \ldots, d'_n) \in \psi_m(n)$ is steeper than d. Then, d can be obtained from d' by a non-empty sequence of elementary flattenings. Without loss of generality, we can assume that d is an elementary flattening of d'. Then, there exist i, j $(i > j)$ such that $d'_i = d_i + 1$ and $d'_j = d_j - 1$, where the other components are the same as for d. i, j can be chosen such that d' is also a decreasing sequence. Now, we prove that $d' \in \hat{\psi}_m(n)$. For a proof by contradiction, assume that this is not the case. It follows that there exists an index k, $1 \le k \le n$, such that $(d'_1, \ldots, d'_k + 1, \ldots, d'_n) \in \psi_m(n)$. Consider the following cases:

1. $k \neq j$
   Flattening the sequence through the pair $(d_i + 1, d_j - 1)$ will lead to $(d_1, \ldots, d_k + 1, \ldots, d_n)$, which belongs to $\psi_m(n)$ by Theorem 5. However, this is greater than d, which is a contradiction.
2. $k = j$
   $(d_1, \ldots, d_i + 1, \ldots, d_j, \ldots, d_n) \in \psi_m(n)$—this is a contradiction. □

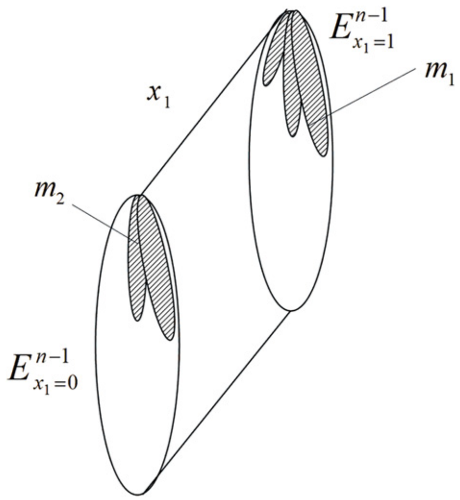

Fig. 5. Composition by the initial segments of the reverse lexicographic ordering.

Theorem 6 implies that if some level of Ξ^n_{m+1} contains more than one element of $\hat{\psi}_m(n)$, then it is sufficient to find the steepest sequences only; if among the steepest sequences of $\psi_m(n)$, there are both upper and non-upper elements, then only the upper sequences must be considered.

4. New classes of upper steepest degree sequences

In this section, we consider the issue of generating new classes of upper steepest degree sequences, where we construct them from the known classes of dimension $n-1$.

Consider $\mathcal{H}_{rmin}$, the class of simple hypergraphs with rank $r_{\min}$. By Theorem 2, $d(\mathcal{H}_{rmin}) \subseteq \hat{\psi}_m(n)$. On the other hand, by Property $P5$, $d(\mathcal{H}_{rmin})$ is a class of steepest sequences. Thus, we obtain an example class from $\hat{\psi}^S_m(n)$.

Consider an arbitrary integer partition of m: $m = m_1 + m_2$, with the only restriction that $2^{n-1} \geq m_1 \geq m_2$. We split E^n according to the value of some x_i (let it be x_1) and consider $R_{m_1}(n-1)$ and $R_{m_2}(n-1)$, the initial m_1 and m_2 reverse lexicographic sequences in $E^{n-1}_{x_1=1}$ and $E^{n-1}_{x_1=0}$.

Theorem 7. *Let $(d'_2, \ldots, d'_n)$ and $(d''_2, \ldots, s''_d)$ be degree sequences of hypergraphs on $[n-1]$ whose edges are represented by $R_{m_1}(n-1)$ and $R_{m_2}(n-1)$, respectively. Let $\mathcal{E}$ denote their union in E^n. Then, $(d_1, \ldots, d_n) = (m_1, d'_2 + d''_2, \ldots, d'_n + d''_n)$, the degree sequence of the hypergraph on $[n]$ whose edges are represented by $\mathcal{E}$, belongs to $\hat{\psi}_m(n)$.*

Proof. It is easy to check that $\mathcal{E}$ is the set of m values 1 for some monotone Boolean function. A geometrical visualisation is given in Fig. 5.

By Property $P5$, $(d'_2, \ldots, d'_n)$ and $(d''_2, \ldots, d''_n)$ are steepest sequences, and by Theorem 3, they belong to $\hat{\psi}_{m_1}(n-1)$ and $\hat{\psi}_{m_2}(n-1)$, respectively. Now we prove that $(d_1, \ldots, d_n) \in \hat{\psi}_m(n)$. Suppose for the purpose of proof by contradiction that $(\tilde{d}_1, \ldots, \tilde{d}_n) > (d_1, \ldots, d_n)$, where $(\tilde{d}_1, \ldots, \tilde{d}_n)$ is the degree sequence of some simple hypergraph on $[n]$. Let $\tilde{\mathcal{E}} \subseteq E^n$ $(|\tilde{\mathcal{E}}| = m)$ represent the set of edges of this hypergraph. Split E^n according to the value of x_1. $\tilde{\mathcal{E}}$ will be partitioned into $\tilde{d}_1$ and $(m - \tilde{d}_1)$-sets in $E^{n-1}_{x_1=1}$ and $E^{n-1}_{x_1=0}$, respectively. These sets can be considered as edges of some hypergraphs on $[n-1]$, and let $(\tilde{d}'_2, \ldots, \tilde{d}'_n)$ and $(\tilde{d}''_2, \ldots, \tilde{d}''_n)$ denote the corresponding degree sequences. Consider the following cases:

(1) $\tilde{d}_1 = d_1$.

Let $\tilde{d}_2 = d_2, \ldots, \tilde{d}_{i-1} = d_{i-1}$, where i is the first index for which $\tilde{d}_i > d_i$. Then, $\tilde{d}'_j = d'_j$ and $\tilde{d}''_j = d''_j$ for all $j < i$; otherwise $\tilde{d}'_j > d'_j$ would imply that $(\tilde{d}'_2, \ldots, \tilde{d}'_n) \notin \psi_{m_1}(n-1)$ and $\tilde{d}''_j > d''_j$ would imply that $(\tilde{d}''_2, \ldots, \tilde{d}''_n) \notin \psi_{m_2}(n-1)$ (by Property $P5$). $\tilde{d}_i > d_i$ implies that either $\tilde{d}'_i > d'_i$ or $\tilde{d}''_i > d''_i$ which leads to the contradiction.

(2) $\tilde{d}_1 > d_1$.

We move $\tilde{d}_1 - d_1$ vertices of $\tilde{\mathcal{E}}$ from $E^{n-1}_{x_1=1}$ to $E^{n-1}_{x_1=0}$. This is possible because there are $\tilde{d}_1 - (m - \tilde{d}_1)$ extra vertices in $E^{n-1}_{x_1=1}$. This operation does not change the sizes of the partitions in any other direction. Now, by the same reasoning as in the previous case, we obtain $(\tilde{d}'_2, \ldots, \tilde{d}'_n) = (d'_2, \ldots, d'_n)$ and $(\tilde{d}''_2, \ldots, \tilde{d}''_n) = (d''_2, \ldots, d''_n)$. According to Theorem 3, $(\tilde{d}'_2, \ldots, \tilde{d}'_n)$ and $(\tilde{d}''_2, \ldots, \tilde{d}''_n)$ are degree sequences of the initial reverse lexicographic sequences in $E^{n-1}_{x_1=1}$ and $E^{n-1}_{x_1=0}$. This is a contradiction. □

Theorem 8. *Let $(d'_2, \dots, d'_n)$ and $(d''_2, \dots, d''_n)$ be degree sequences of hypergraphs on $[n-1]$ whose edges are represented by $R_{m_1}(n-1)$ and $R_{m_2}(n-1)$, respectively. Let $\mathcal{E}$ denote their union in E^n. Then, $(d_1, \dots, d_n) = (m_1, d'_2 + d''_2, \dots, d'_n + d''_n)$, the degree sequence of the hypergraph on $[n]$ whose edges are represented by $\mathcal{E}$, is a steepest sequence.*

Proof. We observe that $(d'_2, \dots, d'_n)$ and $(d''_2, \dots, d''_n)$ are steepest sequences (by Property $P5$), and therefore it is not possible to make $(m_1 = d_1, d_2, \dots, d_n)$ steeper within the set of coordinates $d_2, \dots, d_n$. Now, we prove that it is not possible to make the sequence steeper by using pairs (d_1, d_i). We split E^n according to the values of x_1 and x_i simultaneously: $\mathcal{E}_{x_1=1,x_i=1}$, $\mathcal{E}_{x_1=1,x_i=0}$, $\mathcal{E}_{x_1=0,x_i=1}$, and $\mathcal{E}_{x_1=0,x_i=0}$ are parts of $\mathcal{E}$ in $E^{n-2}_{x_1=1,x_i=1}$, $E^{n-2}_{x_1=1,x_i=0}$, $E^{n-2}_{x_1=0,x_i=1}$, and $E^{n-2}_{x_1=0,x_i=0}$, respectively. Then, $|\mathcal{E}_{x_1=1,x_i=1}| + |\mathcal{E}_{x_1=1,x_i=0}| = d_1$ and $|\mathcal{E}_{x_1=1,x_i=1}| + |\mathcal{E}_{x_1=0,x_i=1}| = d_i$.

First, we observe that if $m_1 \le 2^{n-2}$, then $\mathcal{E}_{x_1=1,x_i=0}$ is empty. If $m_1 > 2^{n-2}$, then $\mathcal{E}_{x_1=1,x_i=1}$ coincides with $E^{n-2}_{x_1=1,x_i=1}$ and $\mathcal{E}_{x_1=0,x_i=1}$ is the initial $(m_1 - 2^{n-2})$-reverse lexicographic sequence in $E^{n-2}_{x_1=1,x_i=0}$ (by Property $P4$). Similarly, if $m_2 \le 2^{n-2}$, then $\mathcal{E}_{x_0=1,x_i=0}$ is empty. If $m_2 > 2^{n-2}$, then $\mathcal{E}_{x_0=1,x_i=1}$ coincides with $E^{n-2}_{x_0=1,x_i=1}$ and $\mathcal{E}_{x_1=0,x_i=0}$ is the initial $(m_2 - 2^{n-2})$-reverse lexicographic sequence in $E^{n-2}_{x_1=0,x_i=0}$ (by Property $P4$). Consider the following cases:

(1) $d_1 < d_i$. It follows that $|\mathcal{E}_{x_1=1,x_i=0}| < |\mathcal{E}_{x_1=0,x_i=1}|$. To make the sequence steeper through the pair (d_1, d_i), we must increase d_i and decrease d_1. This can be performed by moving a vertex from $\mathcal{E}_{x_1=1,x_i=0}$ to $\mathcal{E}_{x_1=0,x_i=1}$. This is not possible because either $\mathcal{E}_{x_1=1,x_i=0}$ is empty or both $\mathcal{E}_{x_1=1,x_i=0}$ and $\mathcal{E}_{x_1=0,x_i=1}$ are initial reverse lexicographic sequences, and $|\mathcal{E}_{x_1=1,x_i=0}| < |\mathcal{E}_{x_1=0,x_i=1}|$.
(2) $d_1 > d_i$. It follows that $|\mathcal{E}_{x_1=1,x_i=0}| > |\mathcal{E}_{x_1=0,x_i=1}|$. To make the sequence steeper, we must increase d_1 and decrease d_i. This can be performed by moving a vertex from $\mathcal{E}_{x_1=0,x_i=1}$ to $\mathcal{E}_{x_1=1,x_i=0}$. This is not possible by the same reasoning. □

If we apply the above theorems to each feasible pair (m_1, m_2), we obtain the following theorem.

Theorem 9. *For any $\tilde{m}$, $]m/2[\le \tilde{m} \le m$ there exists an upper steepest degree sequence in $\hat{\psi}^S_m(n)$ that has a component equal to $\tilde{m}$.*

Thus, we have generated a new class of upper steepest degree sequences. The components of degree sequences are also defined.

We conclude by considering an example that illustrates the results. Consider again the example from Section 3: $n = 5, m = 14$. We compose all feasible pairs of sets in $E^4_{x_1=1}$ and $E^4_{x_1=0}$ and apply the constructions of the above theorems.

Consider all integer partitions of 14.

(1) $m_1 = 13$ and $m_2 = 1$.

$$d' = (8, 8, 7, 7), \qquad d'' = (1, 1, 1, 1) \quad \text{and} \quad d = (13, 9, 9, 8, 8).$$

(2) $m_1 = 12$ and $m_2 = 2$.

$$d' = (8, 8, 6, 6), \qquad d'' = (2, 2, 2, 1), \quad \text{and} \quad d = (12, 10, 10, 8, 7).$$

(3) $m_1 = 11$ and $m_2 = 3$.

$$d' = (8, 7, 6, 6), \qquad d'' = (3, 3, 2, 2), \quad \text{and} \quad d = (11, 11, 10, 8, 8).$$

(4) $m_1 = 10$ and $m_2 = 4$.

$$d' = (8, 6, 6, 5), \qquad d'' = (4, 4, 2, 2), \quad \text{and} \quad d = (10, 12, 10, 8, 7).$$

(5) $m_1 = 9$ and $m_2 = 5$.

$$d' = (8, 5, 5, 5), \qquad d'' = (5, 4, 3, 3), \quad \text{and} \quad d = (9, 13, 9, 8, 8).$$

(6) $m_1 = 8$ and $m_2 = 6$.

$$d' = (8, 4, 4, 4), \qquad d'' = (6, 4, 4, 3), \quad \text{and} \quad d = (8, 14, 8, 8, 7).$$

(7) $m_1 = 7$ and $m_2 = 7$.

$$d' = (7, 4, 4, 4), \qquad d'' = (7, 4, 4, 4), \quad \text{and} \quad d = (7, 14, 8, 8, 8).$$

Thus, we get the following class of upper steepest degree sequences (presented in decreasing order):

$$\{(14, 8, 8, 8, 7), (13, 9, 9, 8, 8), (12, 10, 10, 8, 7), (11, 11, 10, 8, 8)\}.$$

While $\hat{\psi}^S_{14}(5) = \{(14, 8, 8, 8, 7), (13, 9, 9, 8, 8), (12, 10, 10, 8, 7), (12, 10, 9, 9, 8), (11, 11, 10, 8, 8), (11, 10, 10, 9, 9)\}$.

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