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Note

Interval edge-colorings of composition of graphs

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ABSTRACT

An edge-coloring of a graph G with consecutive integers $c_1, \dots, c_t$ is called an *interval t-coloring* if all colors are used, and the colors of edges incident to any vertex of G are distinct and form an interval of integers. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. In 2004, Giaro and Kubale showed that if $G, H \in \mathfrak{N}$, then the Cartesian product of these graphs belongs to $\mathfrak{N}$. In the same year they formulated a similar problem for the composition of graphs as an open problem. Later, in 2009, the second author showed that if $G, H \in \mathfrak{N}$ and H is a regular graph, then $G[H] \in \mathfrak{N}$. In this paper, we prove that if $G \in \mathfrak{N}$ and H has an interval coloring of a special type, then $G[H] \in \mathfrak{N}$. Moreover, we show that all regular graphs, complete bipartite graphs and trees have such a special interval coloring. In particular, this implies that if $G \in \mathfrak{N}$ and T is a tree, then $G[T] \in \mathfrak{N}$.

1. Introduction

All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. For a graph G, by $\overline{G}$ we denote the complement of the graph G. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of G by $\Delta(G)$, and the chromatic index of G by $\chi'(G)$. The terms and concepts that we do not define can be found in [1,8,20,34].

A proper edge-coloring of a graph G is a coloring of the edges of G such that no two adjacent edges receive the same color. A proper edge-coloring of a graph G with consecutive integers $c_1, \dots, c_t$ is an *interval t-coloring* if all colors are used, and the colors of edges incident to each vertex of G form an interval of integers. A graph G is *interval colorable* if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. The concept of interval edge-coloring of graphs was introduced by Asratian and Kamalian [2] in 1987. In [2], they proved that if $G \in \mathfrak{N}$, then $\chi'(G) = \Delta(G)$. Asratian and Kamalian also proved [2,3] that if a triangle-free graph G admits an interval t-coloring, then $t \leq |V(G)| - 1$. In [16,17], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved that the complete bipartite graph $K_{m,n}$ has an interval t-coloring if and only if $m + n - \gcd(m, n) \leq t \leq m + n - 1$, where $\gcd(m, n)$ is the greatest common divisor of m and n. In [24], Petrosyan investigated interval colorings of complete graphs and hypercubes. In particular, he proved that if $n \leq t \leq \frac{n(n+1)}{2}$, then the hypercube Q_n has an interval t-coloring. Later, in [27], it was shown that the hypercube Q_n has an interval t-coloring if and only if $n \leq t \leq \frac{n(n+1)}{2}$. In [31], Sevast'janov

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proved that it is an *NP*-complete problem to decide whether a bipartite graph has an interval coloring or not. In papers [2,3,6,7,9,16,17,20,24,26–28,31], the problems of the existence, construction and estimating of the numerical parameters of interval colorings of graphs were investigated. Surveys on this topic can be found in some books [1,15,20].

Graph products [8] were first introduced by Berge [5], Sabidussi [30], Harary [10] and Vizing [32]. In particular, Sabidussi [30] and Vizing [32] showed that every connected graph has a unique decomposition into prime factors with respect to the Cartesian product. In the same direction there are also many interesting problems of decomposing of the different products of graphs into Hamiltonian cycles. In particular, in [4] it was proved Bermond's conjecture that states: if two graphs are decomposable into Hamiltonian cycles, then their composition is decomposable, too. A lot of work was done on various topics related to graph products, on the other hand there are still many questions open. For example, it is still open Hedetniemi's conjecture [12], Vizing's conjecture [33] and the conjecture of Harary, Kainen and Schwenk [11].

There are many papers [13,14,19,21–23,29,35] devoted to proper edge-colorings of various products of graphs, however very little is known on interval colorings of graph products. Interval colorings of Cartesian products of graphs were first investigated by Giaro and Kubale [6]. In [7], Giaro and Kubale proved that if $G, H \in \mathfrak{N}$, then $G \square H \in \mathfrak{N}$. In 2004, they formulated [20] a similar problem for the composition of graphs as an open problem. In 2009, the second author [25] showed that if $G, H \in \mathfrak{N}$ and H is a regular graph, then $G[H] \in \mathfrak{N}$. Later, Yepremyan [28] proved that if G is a tree and H is either a path or a star, then $G[H] \in \mathfrak{N}$. Some other results on interval colorings of various products of graphs were obtained in [20,25–28].

In this paper, we prove that if $G \in \mathfrak{N}$ and H has an interval coloring of a special type, then $G[H] \in \mathfrak{N}$. Moreover, we show that all regular graphs, complete bipartite graphs and trees have such a special interval coloring. In particular, this implies that if $G \in \mathfrak{N}$ and T is a tree, then $G[T] \in \mathfrak{N}$.

2. Notations, definitions and auxiliary results

We use standard notations C_n and K_n for the simple cycle and complete graph on n vertices, respectively. We also use standard notations $K_{m,n}$ and $K_{m,n,l}$ for the complete bipartite and tripartite graphs, respectively, one part of which has m vertices, the other part has n vertices and the third part has l vertices.

For two positive integers a and b with $a \leq b$, we denote by $[a, b]$ the interval of integers $\{a, \ldots, b\}$.

Let $L = (l_1, \ldots, l_k)$ be an ordered sequence of nonnegative integers. The smallest and largest elements of L are denoted by $\underline{L}$ and $\overline{L}$, respectively. The length (the number of elements) of L is denoted by $|L|$. By $L(i)$, we denote the ith element of L $(1 \leq i \leq k)$. An ordered sequence $L = (l_1, \ldots, l_k)$ is called a *continuous sequence* if it contains all integers between $\underline{L}$ and $\overline{L}$. If $L = (l_1, \ldots, l_k)$ is an ordered sequence and p is nonnegative integer, then the sequence $(l_1 + p, \ldots, l_k + p)$ is denoted by $L \oplus p$. Clearly, $(L \oplus p)(i) = L(i) + p$ for any $p \in \mathbb{Z}_+$.

Let G and H be two graphs. The composition (lexicographic product) $G[H]$ of graphs G and H is defined as follows:

$$V(G[H]) = V(G) \times V(H),$$
$$E(G[H]) = \left\{(u_1, v_1)(u_2, v_2) \colon u_1u_2 \in E(G) \vee (u_1 = u_2 \wedge v_1v_2 \in E(H))\right\}.$$

A *partial edge-coloring* of G is a coloring of some of the edges of G such that no two adjacent edges receive the same color. If α is a proper edge-coloring of G and $v \in V(G)$, then $S(v, \alpha)$ (*spectrum* of a vertex v) denotes the set of all colors appearing on edges incident to v. The smallest and largest colors of $S(v, \alpha)$ are denoted by $\underline{S}(v, \alpha)$ and $\overline{S}(v, \alpha)$, respectively. A proper edge-coloring α of G with consecutive integers $c_1, \ldots, c_t$ is called an *interval t-coloring* if all colors are used, and for any $v \in V(G)$, the set $S(v, \alpha)$ is an interval of integers. A graph G is *interval colorable* if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the smallest and the largest values of t for which it has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively.

In [2,3], Asratian and Kamalian obtained the following result.

Theorem 1. *If $G \in \mathfrak{N}$, then $\chi'(G) = \Delta(G)$. Moreover, if G is a regular graph, then $G \in \mathfrak{N}$ if and only if $\chi'(G) = \Delta(G)$.*

In [16], Kamalian proved the following result on complete bipartite graphs.

Theorem 2. *For any $m, n \in \mathbb{N}$, the complete bipartite graph $K_{m,n}$ is interval colorable, and*

(1) $w\left(K_{m,n}\right) = m + n - \gcd(m, n)$,
(2) $W\left(K_{m,n}\right) = m + n - 1$,
(3) *if $w\left(K_{m,n}\right) \leq t \leq W\left(K_{m,n}\right)$, then $K_{m,n}$ has an interval t-coloring.*

In [18], König proved the following result on bipartite graphs.

Theorem 3. *If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.*

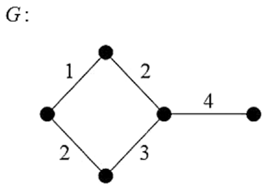

Fig. 1. The graph G with its coloring α and with $LSE(V(G),\alpha)=(1,1,2,2,4)$, $USE(V(G),\alpha)=(2,2,3,4,4)$.

Let α be a proper edge-coloring of G and $V'=\{v_1,\dots,v_k\}\subseteq V(G)$. We define two ordered sequences $LSE(V',\alpha)$ (*Lower Spectral Edge*) and $USE(V',\alpha)$ (*Upper Spectral Edge*) as follows:

$$LSE(V',\alpha)=\left(\underline{S}\left(v_{i_1},\alpha\right),\underline{S}\left(v_{i_2},\alpha\right),\dots,\underline{S}\left(v_{i_k},\alpha\right)\right),$$

where $\underline{S}\left(v_{i_l},\alpha\right)\le\underline{S}\left(v_{i_{l+1}},\alpha\right)$ for $1\le l\le k-1$, and

$$USE(V',\alpha)=\left(\overline{S}\left(v_{j_1},\alpha\right),\overline{S}\left(v_{j_2},\alpha\right),\dots,\overline{S}\left(v_{j_k},\alpha\right)\right),$$

where $\overline{S}\left(v_{j_l},\alpha\right)\le\overline{S}\left(v_{j_{l+1}},\alpha\right)$ for $1\le l\le k-1$.

For example, if we consider the graph G with its coloring α shown in Fig. 1, then $LSE(V(G),\alpha)=(1,1,2,2,4)$ and $USE(V(G),\alpha)=(2,2,3,4,4)$. Moreover, the sequence $(1,1,2,2,4)$ is not continuous, but the sequence $(2,2,3,4,4)$ is continuous.

Recall that for ordered sequences $LSE(V',\alpha)$ and $USE(V',\alpha)$, the number of elements in $LSE(V',\alpha)$ and $USE(V',\alpha)$ is denoted by $|LSE(V',\alpha)|$ and $|USE(V',\alpha)|$, respectively. Clearly, $|LSE(V(G),\alpha)|=|USE(V(G),\alpha)|=|V(G)|$.

We also need the following lemma.

Lemma 4. *If $K_{n,n}$ is a complete bipartite graph with bipartition (U,V), then for any continuous sequence L with length n, $K_{n,n}$ has an interval coloring α such that*

$$LSE(U,\alpha)=LSE(V,\alpha)=L.$$

Proof. Let $K_{n,n}$ be a complete bipartite graph with bipartition (U,V), where $U=\{u_1,\dots,u_n\}$ and $V=\{v_1,\dots,v_n\}$. Also, let $L=\left(\underbrace{l_1,\dots,l_1}_{n_1},\underbrace{l_2,\dots,l_2}_{n_2},\dots,\underbrace{l_k,\dots,l_k}_{n_k}\right)$ be a continuous sequence with length n $\left(\sum_{i=1}^{k}n_i=n\right)$. Clearly, $l_{i+1}=l_i+1$ for $1\le l\le k-1$.

First we define a partial edge-coloring α of $K_{n,n}$ as follows:

(1) for $1\le i\le k-1$ and $p+q=1+\sum_{j=1}^{i}n_j$, let $\alpha\left(u_pv_q\right)=l_i$;
(2) for $1\le i\le k-1$ and $p+q=n+1+\sum_{j=1}^{i}n_j$, let $\alpha\left(u_pv_q\right)=l_i+n$,

where $p,q\in[1,n]$.

Define a subgraph G of $K_{n,n}$ as follows:

$$V(G)=V(K_{n,n})\quad\text{and}\quad E(G)=\left\{e\colon e\in E(K_{n,n})\wedge\alpha(e)\in[l_1,l_{k-1}]\cup[l_1+n,l_{k-1}+n]\right\}.$$

By the definition of α, G is a spanning $(k-1)$-regular bipartite subgraph of $K_{n,n}$. Next we define a subgraph G' of $K_{n,n}$ as follows:

$$V(G)=V(K_{n,n})\quad\text{and}\quad E\left(G'\right)=E\left(K_{n,n}\right)\setminus E(G).$$

Clearly, G' is a spanning $(n-k+1)$-regular bipartite subgraph of $K_{n,n}$. By Theorem 3, $\chi'\left(G'\right)=\Delta\left(G'\right)=n-k+1$. Let β be a proper edge-coloring of G' with colors $l_k,l_k+1,\dots,l_k+n-k$. By the definition of β, for each vertex $v\in V(K_{n,n})$, $S(v,\beta)=[l_k,l_k+n-k]$.

Now we are able to define an edge-coloring γ of $K_{n,n}$.

For every $e\in E(K_{n,n})$, let

$$\gamma(e)=\begin{cases}\alpha(e), & \text{if } e\in E(G),\\ \beta(e), & \text{if } e\in E\left(G'\right).\end{cases}$$

Let us prove that γ is an interval (l_k+n-1)-coloring of $K_{n,n}$ such that $S(u_i,\gamma)=S(v_i,\gamma)$ and $\underline{S}(u_i,\gamma)=\underline{S}(v_i,\gamma)=l_i$ for $1\le i\le n$.

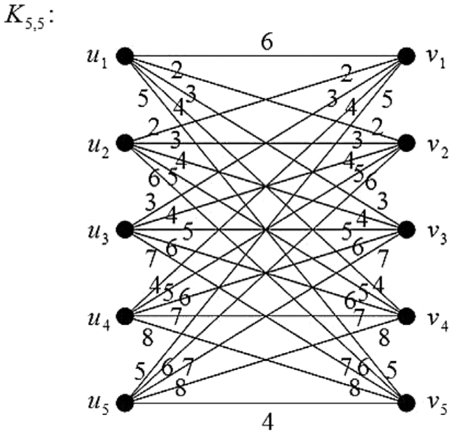

Fig. 2. The interval coloring γ of $K_{5,5}$ with $LSE(U,\gamma) = LSE(V,\gamma) = (2,2,3,4,4)$ that is described in the proof of Lemma 4.

By the definition of γ, for $1 \le i \le n$, we have

$$\begin{aligned} &S(u_i,\gamma) = S(v_i,\gamma) = [l_1, l_1 + n - 1] \quad \text{if } i \in [1, n_1],\\ &S(u_i,\gamma) = S(v_i,\gamma) = [l_2, l_2 + n - 1] \quad \text{if } i \in [n_1 + 1, n_1 + n_2],\\ &\ldots\\ &S(u_i,\gamma) = S(v_i,\gamma) = [l_k, l_k + n - 1] \quad \text{if } i \in \left[\sum_{j=1}^{k-1} n_j + 1, \sum_{j=1}^{k} n_j\right]. \end{aligned}$$

This implies that γ is an interval $(l_k + n - 1)$-coloring of $K_{n,n}$ and $LSE(U,\gamma) = LSE(V,\gamma) = L$ (see Fig. 2). □

3. The main result

Here, we prove our main result which states that if $G \in \mathfrak{N}$ and H has an interval coloring of a special type, then $G[H] \in \mathfrak{N}$.

Theorem 5. *If $G \in \mathfrak{N}$ and H has an interval coloring α_H such that $USE(V(H),\alpha_H)$ is continuous, then $G[H] \in \mathfrak{N}$. Moreover, if $|V(H)| = n$ and $L = USE(V(H),\alpha_H)$, then*

$$w\left(G[H]\right) \le w(G) \cdot n + \bar{L} \quad \text{and} \quad W\left(G[H]\right) \ge W(G) \cdot n + \bar{L}.$$

Proof. Let $V(G) = \{u_1,\ldots,u_m\}$, $V(H) = \{w_1,\ldots,w_n\}$ and

$$V(G[H]) = \left\{v_j^{(i)} \colon 1 \le i \le m,\ 1 \le j \le n\right\} \quad \text{and}$$

$$E(G[H]) = \left\{v_p^{(i)} v_q^{(j)} \colon u_i u_j \in E(G),\ 1 \le p \le n,\ 1 \le q \le n\right\} \cup \bigcup_{i=1}^{m} E^i,$$

where $E^i = \left\{v_p^{(i)} v_q^{(i)} \colon w_p w_q \in E(H)\right\}$.

Let α_G be an interval t-coloring of G and L be a continuous sequence with length n such that $L = USE(V(H),\alpha_H)$. Without loss of generality we may assume that vertices of H are numbered so that $\overline{S}\left(w_i,\alpha_H\right) = L(i)$ for $1 \le i \le n$. Let us consider the graph $K_2\left[\overline{K}_{|V(H)|}\right]$. Clearly, $K_2\left[\overline{K}_{|V(H)|}\right]$ is isomorphic to $K_{n,n}$. Let $V\left(K_2\left[\overline{K}_{|V(H)|}\right]\right) = \{x_1,\ldots,x_n,y_1,\ldots,y_n\}$ and $E\left(K_2\left[\overline{K}_{|V(H)|}\right]\right) = \left\{x_i y_j \colon 1 \le i \le n,\ 1 \le j \le n\right\}$. Since L is a continuous sequence, $L \oplus 1$ is a continuous sequence, too. By Lemma 4, $K_2\left[\overline{K}_{|V(H)|}\right]$ has an interval coloring β such that $\underline{S}\left(x_i,\beta\right) = \underline{S}\left(y_i,\beta\right) = L(i) + 1$ for $1 \le i \le n$.

Now we are able to define an edge-coloring $\alpha_{G[H]}$ of $G[H]$.

1) For $1 \le i \le m$ and $v_p^{(i)} v_q^{(i)} \in E^i$ $(1 \le p \le n,\ 1 \le q \le n)$, let

$$\alpha_{G[H]}\left(v_p^{(i)} v_q^{(i)}\right) = \left(\underline{S}\left(u_i,\alpha_G\right) - 1\right) n + \alpha_H\left(w_p w_q\right).$$

2) For $1 \le i < j \le m$ and $v_p^{(i)} v_q^{(j)} \in E(G[H])$ $(1 \le p \le n,\ 1 \le q \le n)$, let

$$\alpha_{G[H]}\left(v_p^{(i)} v_q^{(j)}\right) = \left(\alpha_G\left(u_i u_j\right) - 1\right) n + \beta\left(x_p y_q\right).$$

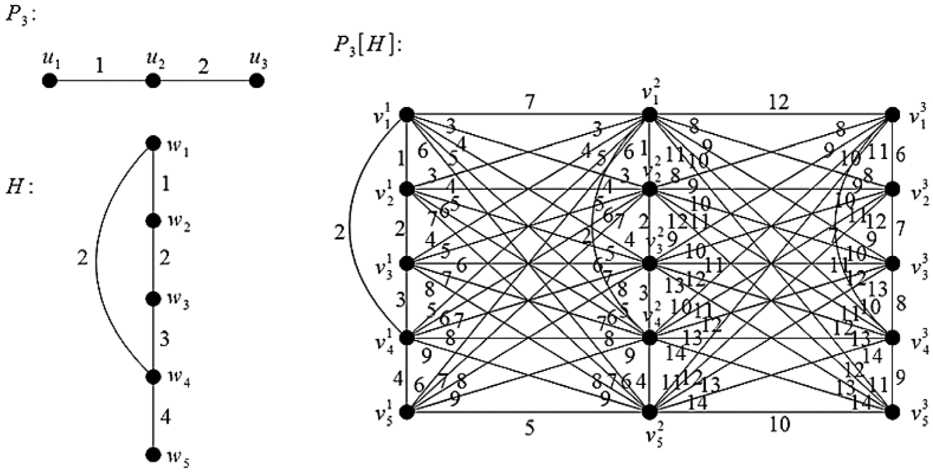

Fig. 3. The interval 14-coloring $\alpha_{P_3[H]}$ of $P_3[H]$ that is described in the proof of Theorem 5.

It is not difficult to see that $\alpha_{G[H]}$ is a proper edge-coloring of $G[H]$. Let us prove that $\alpha_{G[H]}$ is an interval $(t\cdot n+\overline{L})$-coloring of $G[H]$. For the proof, it suffices to show that for $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$\overline{S}\left(v_j^{(i)}, \alpha_{G[H]}\right) - \underline{S}\left(v_j^{(i)}, \alpha_{G[H]}\right) = d_{G[H]}\left(v_j^{(i)}\right) - 1.$$

By the definition of $\alpha_{G[H]}$, for $1 \leq i \leq m$ and $1 \leq j \leq n$, we have

$$\overline{S}\left(v_j^{(i)}, \alpha_{G[H]}\right) = \left(\overline{S}\left(u_i, \alpha_G\right) - 1\right) n + L(j) + 1 + n - 1 = \overline{S}\left(u_i, \alpha_G\right) \cdot n + L(j).$$

By the definition of $\alpha_{G[H]}$ and taking into account that $L(j) - \underline{S}\left(w_j, \alpha_H\right) = d_H\left(w_j\right) - 1\ (1 \leq j \leq n)$, for $1 \leq i \leq m$ and $1 \leq j \leq n$, we have

$$\underline{S}\left(v_j^{(i)}, \alpha_{G[H]}\right) = \left(\underline{S}\left(u_i, \alpha_G\right) - 1\right) n + L(j) - d_H\left(w_j\right) + 1.$$

Now, taking into account that $\overline{S}\left(u_i, \alpha_G\right) - \underline{S}\left(u_i, \alpha_G\right) = d_G(u_i) - 1\ (1 \leq i \leq m)$, for $1 \leq i \leq m$ and $1 \leq j \leq n$, we obtain

$$\begin{aligned} \overline{S}\left(v_j^{(i)}, \alpha_{G[H]}\right) - \underline{S}\left(v_j^{(i)}, \alpha_{G[H]}\right) &= \left(\overline{S}\left(u_i, \alpha_G\right) - \underline{S}\left(u_i, \alpha_G\right) + 1\right) n + d_H\left(w_j\right) - 1 \\ &= d_G\left(u_i\right) \cdot n + d_H\left(w_j\right) - 1 = d_{G[H]}\left(v_j^{(i)}\right) - 1. \end{aligned}$$

This shows that $\alpha_{G[H]}$ is an interval $(t\cdot n+\overline{L})$-coloring of $G[H]$. Thus, $w\left(G[H]\right) \leq w(G)\cdot n+\overline{L}$ and $W\left(G[H]\right) \geq W(G)\cdot n+\overline{L}$ (see Fig. 3). □

Corollary 6. *If* $G, H \in \mathfrak{N}$ *and* H *is an* r*-regular graph, then* $G[H] \in \mathfrak{N}$*. Moreover, if* $|V(H)| = n$*, then*

$$w(G[H]) \leq w(G)\cdot n + r \quad \text{and} \quad W(G[H]) \geq W(G)\cdot n + r.$$

Proof. Since $H \in \mathfrak{N}$ and H is an r-regular graph, by Theorem 1, $\chi'(H) = \Delta(H) = r$. This implies that H has a proper edge-coloring α_H with colors $1, \ldots, r$. Hence, for every $v \in V(H), S\left(v, \alpha_H\right) = [1, r]$. Clearly, α_H is an interval r-coloring and $USE(V(H), \alpha_H) = (r, \ldots, r)$ is continuous, so, by Theorem 5, $G[H] \in \mathfrak{N}$. Moreover, if $|V(H)| = n$, then $w(G[H]) \leq w(G)\cdot n+r$ and $W(G[H]) \geq W(G)\cdot n + r$. □

Corollary 7. *Let* $n \in \mathbb{N}$*. If* $G \in \mathfrak{N}$*, then* $G[\overline{K}_n] \in \mathfrak{N}$ *and moreover we have* $w(G[\overline{K}_n]) \leq w(G)\cdot n$ *and* $W(G[\overline{K}_n]) \geq W(G)\cdot n$*.*

Proof. We may assume that $\overline{K}_n$ has an interval coloring α such that $USE(V(\overline{K}_n), \alpha) = (0, \ldots, 0)$. Since $USE(V(\overline{K}_n), \alpha) = (0, \ldots, 0)$ is continuous, by Theorem 5, $G[\overline{K}_n] \in \mathfrak{N}$. Moreover, $w(G[\overline{K}_n]) \leq w(G)\cdot n$ and $W(G[\overline{K}_n]) \geq W(G)\cdot n$. □

4. Applications of the main result

This section is devoted to applications of the main result from the previous section for some classes of graphs. We first consider complete bipartite graphs.

Theorem 8. *Let* $m, n \in \mathbb{N}$*. If* $G \in \mathfrak{N}$*, then* $G[K_{m,n}] \in \mathfrak{N}$ *and moreover we have*

$$w\left(G[K_{m,n}]\right) \leq (w(G)+1)(m+n)-1 \quad \text{and} \quad W\left(G[K_{m,n}]\right) \geq (W(G)+1)(m+n)-1.$$

Proof. Let (U, V) be a bipartition of $K_{m,n}$, where $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$. Define an edge-coloring α of $K_{m,n}$ as follows: for each edge $u_iv_j \in E(K_{m,n})$, let $\alpha(u_iv_j) = i+j-1$, where $1 \leq i \leq m$, $1 \leq j \leq n$. Clearly, α is an interval $(m+n-1)$-coloring of $K_{m,n}$. Moreover, $S(u_i, \alpha) = [i, i+n-1]$ for $1 \leq i \leq m$ and $S(v_j, \alpha) = [j, j+m-1]$ for $1 \leq j \leq n$. This implies that $USE(U, \alpha) = (n, n+1, \ldots, m+n-1)$ and $USE(V, \alpha) = (m, m+1, \ldots, m+n-1)$. Since $USE(V\left(K_{m,n}\right), \alpha)$ is the union of $USE(U, \alpha)$ and $USE(V, \alpha)$, we obtain $USE(V\left(K_{m,n}\right), \alpha)$ is a continuous sequence. By Theorem 5, $G[K_{m,n}] \in \mathfrak{N}$. Moreover, $w(G[K_{m,n}]) \leq w(G) \cdot (m+n) + m+n-1$ and $W(G[K_{m,n}]) \geq W(G) \cdot (m+n) + m+n-1$. □

Next, we consider complete graphs of even order. Here we need one result on interval colorings of complete graphs of even order. In [24], it was proved the following result.

Theorem 9 *([24]). Let* $n \in \mathbb{N}$*. Then* K_{2n} *has an interval* $(3n-2)$*-coloring* α *such that for each* $i \in [1, n]$*, there are two different vertices* $x_i, y_i \in V(K_{2n})$ *such that* $\underline{S}(x_i, \alpha) = \underline{S}(y_i, \alpha) = i$*.*

Now we are able to prove our result on complete graphs of even order.

Theorem 10. *Let* $n \in \mathbb{N}$*. If* $G \in \mathfrak{N}$*, then* $G[K_{2n}] \in \mathfrak{N}$ *and moreover we have*

$$w\left(G[K_{2n}]\right) \leq (2 \cdot w(G)+2)n-1 \quad \text{and} \quad W\left(G[K_{2n}]\right) \geq (2 \cdot W(G)+3)n-2.$$

Proof. By Corollary 6, if $G \in \mathfrak{N}$, then $G[K_{2n}] \in \mathfrak{N}$ and $w\left(G[K_{2n}]\right) \leq w(G) \cdot 2n + 2n - 1$.

Now we show that $W\left(G[K_{2n}]\right) \geq (2 \cdot W(G)+3)n-2$. By Theorem 9, K_{2n} has an interval $(3n-2)$-coloring α such that for each $i \in [1, n]$, there are two different vertices $x_i, y_i \in V(K_{2n})$ such that $S(x_i, \alpha) = S(y_i, \alpha) = [i, i+2n-2]$. This implies that $USE(V(K_{2n}), \alpha) = (2n-1, 2n-1, 2n, 2n, \ldots, 3n-2, 3n-2)$, which is a continuous sequence. By Theorem 5, $G[K_{2n}] \in \mathfrak{N}$ and $W(G[K_{2n}]) \geq W(G) \cdot 2n + 3n - 2$. □

A similar result also can be obtained for even cycles.

Theorem 11. *Let* $n \in \mathbb{N}$ *and* $n \geq 2$*. If* $G \in \mathfrak{N}$*, then* $G[C_{2n}] \in \mathfrak{N}$ *and moreover we have*

$$w\left(G[C_{2n}]\right) \leq 2(w(G) \cdot n + 1) \quad \text{and} \quad W\left(G[C_{2n}]\right) \geq (2 \cdot W(G)+1)n+1.$$

Proof. By Corollary 6, if $G \in \mathfrak{N}$, then $G[C_{2n}] \in \mathfrak{N}$ and $w\left(G[C_{2n}]\right) \leq w(G) \cdot 2n + 2$.

Now we show that $W\left(G[C_{2n}]\right) \geq (2 \cdot W(G)+1)n+1$. Let $V(C_{2n}) = \{v_1, \ldots, v_{2n}\}$ and $E(C_{2n}) = \{v_iv_{i+1} \colon 1 \leq i \leq 2n-1\} \cup \{v_1v_{2n}\}$. Define an edge-coloring α of C_{2n} as follows: for $1 \leq i \leq n$, let $\alpha(v_iv_{i+1}) = \alpha(v_{2n+1-i}v_{2n-i}) = i+1$ and $\alpha(v_1v_{2n}) = 1$. Clearly, α is an interval $(n+1)$-coloring of C_{2n} such that for each $i \in [1, n]$, $S\left(v_i, \alpha\right) = S\left(v_{2n+1-i}, \alpha\right) = [i, i+1]$. This implies that $USE(V(C_{2n}), \alpha) = (2, 2, 3, 3, \ldots, n+1, n+1)$, which is a continuous sequence. By Theorem 5, $G[C_{2n}] \in \mathfrak{N}$ and $W(G[C_{2n}]) \geq W(G) \cdot 2n + n + 1$. □

Finally, we show that every tree T has an interval coloring α such that $USE(V(T), \alpha)$ is continuous.

Theorem 12. *If* T *is a tree, then it has an interval coloring* α *such that* $USE(V(T), \alpha)$ *is continuous.*

Proof. Let T be a tree with $|V(T)| = n$ $(n \geq 2)$. We prove the theorem by induction on $|E(T)|$. We will construct tree T starting from some v_1v_2 edge and adding a new leaf on each step. For $1 \leq i \leq n-1$, we denote by T_i the tree obtained on step i and by α_i its edge-coloring. For a tree T_i and its edge-coloring α_i $(1 \leq i \leq n-1)$, define numbers a_i and b_i as follows:

$$a_i = \min_{e \in E(T_i)} \alpha_i(e) \quad \text{and} \quad b_i = \max_{e \in E(T_i)} \alpha_i(e).$$

We show that in each step T_i and α_i satisfy the following two conditions:

(1) for each $v \in V(T_i)$, $S\left(v, \alpha_i\right)$ is an interval of integers;
(2) each color of the interval $[a_i, b_i]$ appears in $USE\left(V(T_i), \alpha_i\right)$.

Let $V(T_1) = \{v_1, v_2\}$ and $E(T_1) = \{v_1v_2\}$. Define an edge-coloring α_1 of T_1 as follows: $\alpha_1(v_1v_2) = |E(T)|$. Since $S\left(v_1, \alpha_1\right) = S\left(v_2, \alpha_1\right) = \{|E(T)|\}$, we have $a_1 = b_1 = |E(T)|$ and $USE\left(V(T_1), \alpha_1\right) = (|E(T)|, |E(T)|)$. This implies that (1) and (2) hold for T_1. Suppose that $n \geq 3$, (1) and (2) are satisfied for a tree T_{m-1} and its edge-coloring α_{m-1}, and prove that (1) and (2) are also satisfied for a tree T_m and its edge-coloring α_m $(2 \leq m \leq n-1)$. Let u be the pendant vertex that should be added to T_{m-1} to get T_m. Also, let $uw \in E(T_m)$, where $w \in V(T_{m-1})$.

Define an edge-coloring α_m of T_m as follows: for every $e \in E(T_m)$, let

$$\alpha_m(e) = \begin{cases} \alpha_{m-1}(e), & \text{if } e \in E(T_{m-1}), \\ \underline{S}\left(w, \alpha_{m-1}\right) - 1, & \text{if } e = uw. \end{cases}$$

By the definition of α_m, we have:

1) for each $v \in V(T_m)$, $S(v, \alpha_m)$ is an interval of integers;
2) for $v \in V(T_{m-1})$, $\overline{S}(v, \alpha_m) = \overline{S}(v, \alpha_{m-1})$ and $USE(V(T_m), \alpha_m)$ is the union of $USE(V(T_{m-1}), \alpha_{m-1})$ and $(\alpha_m(uw))$;
3) $a_m = \min\{a_{m-1}, \alpha_m(uw)\}$, $b_m = b_{m-1}$ and $\alpha_m(uw) = \underline{S}(w, \alpha_{m-1}) - 1 \geq a_{m-1} - 1$.

By 1), 2) and 3), and taking into account that each color of the interval $[a_{m-1}, b_{m-1}]$ appears in $USE(V(T_{m-1}), \alpha_{m-1})$, we obtain that each color of the interval $[a_m, b_m]$ appears in $USE(V(T_m), \alpha_m)$. This implies that (1) and (2) also hold for T_m. So, taking $m = n - 1$, we get that $T = T_{n-1}$. Finally, define an edge-coloring α of T as follows: for every $e \in E(T)$, let $\alpha(e) = \alpha_{n-1}(e) - a_{n-1} + 1$. It is not difficult to see that α is an interval $(|E(T)| - a_{n-1} + 1)$-coloring of T such that $USE(V(T), \alpha)$ is continuous. □

Corollary 13. *If $G \in \mathfrak{N}$ and T is a tree, then $G[T] \in \mathfrak{N}$.*

5. Concluding remarks

In the previous sections it was proved that if $G \in \mathfrak{N}$ and H has an interval coloring α_H such that $USE(V(H), \alpha_H)$ is continuous, then $G[H] \in \mathfrak{N}$. Unfortunately, not all interval colorable graphs have such a special interval coloring. For example, if we consider the complete tripartite graph $K_{1,1,2n}$ $(n \geq 2)$, then it is not difficult to see that for every interval coloring α of $K_{1,1,2n}$ $(n \geq 2)$, $USE(V(K_{1,1,2n}), \alpha)$ is not continuous. This implies that the problem on interval colorability of the composition of interval colorable graphs still remains open.

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References

[1] A.S. Asratian, T.M.J. Denley, R. Haggkvist, Bipartite Graphs and their Applications, Cambridge University Press, Cambridge, 1998.
[2] A.S. Asratian, R.R. Kamalian, Interval colorings of edges of a multigraph, Appl. Math. 5 (1987) 25–34. (in Russian).
[3] A.S. Asratian, R.R. Kamalian, Investigation on interval edge-colorings of graphs, J. Combin. Theory Ser. B 62 (1994) 34–43.
[4] Z. Baranyai, Gy.R. Szasz, Hamiltonian decomposition of lexicographic product, J. Combin. Theory Ser. B 31 (1981) 253–261.
[5] C. Berge, Theorie des Graphes et ses Applications, Dunod, Paris, 1958, (in French).
[6] K. Giaro, M. Kubale, Consecutive edge-colorings of complete and incomplete Cartesian products of graphs, Congr, Numer. 128 (1997) 143–149.
[7] K. Giaro, M. Kubale, Compact scheduling of zero–one time operations in multi-stage systems, Discrete Appl. Math. 145 (2004) 95–103.
[8] R. Hammack, W. Imrich, S. Klavzar, Handbook of Product Graphs, second ed., CRC Press, 2011.
[9] H.M. Hansen, Scheduling with minimum waiting periods (MSc Thesis), Odense University, Odense, Denmark, 1992, (in Danish).
[10] F. Harary, Graph Theory, Addison-Wesley, Reading MA, 1969.
[11] F. Harary, P.C. Kainen, A.J. Schwenk, Toroidal graphs with arbitrarily high crossing numbers, Nanta Math. 6 (1973) 58–67.
[12] S. Hedetniemi, Homomorphisms of graphs and automata (Doctoral Dissertation), University of Michigan, 1966.
[13] P.E. Himmelwright, J.E. Williamson, On 1-factorability and edge-colorability of cartesian products of graphs, Elem. Math. 29 (1974) 66–67.
[14] M.M.M. Jaradat, On the Anderson-Lipman conjecture and some related problems, Discrete Math. 297 (2005) 167–173.
[15] T.R. Jensen, B. Toft, Graph Coloring Problems, Wiley Interscience Series in Discrete Mathematics and Optimization, 1995.
[16] R.R. Kamalian, Interval colorings of complete bipartite graphs and trees, preprint, Comp. Cen. of Acad. Sci. of Armenian SSR, Yerevan, 1989 (in Russian).
[17] R.R. Kamalian, Interval edge colorings of graphs (Doctoral Thesis), Novosibirsk, 1990.
[18] D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, Math. Ann. 77 (1916) 453–465.
[19] A. Kotzig, 1-Factorizations of cartesian products of regular graphs, J. Graph Theory 3 (1979) 23–34.
[20] M. Kubale, Graph Colorings, American Mathematical Society, 2004.
[21] E.S. Mahamoodian, On edge-colorability of cartesian products of graphs, Canad. Math. Bull. 24 (1981) 107–108.
[22] B. Mohar, On edge-colorability of products of graphs, Publ. Inst. Math. (Beograd) 36 (50) (1984) 13–16.
[23] B. Mohar, T. Pisanski, Edge-coloring of a family of regular graphs, Publ. Inst. Math. (Beograd) 33 (47) (1983) 157–162.
[24] P.A. Petrosyan, Interval edge-colorings of complete graphs and n-dimensional cubes, Discrete Math. 310 (2010) 1580–1587.
[25] P.A. Petrosyan, Interval edge colorings of some products of graphs, Discuss. Math. Graph Theory 31 (2) (2011) 357–373.
[26] P.A. Petrosyan, G.H. Karapetyan, Lower bounds for the greatest possible number of colors in interval edge colorings of bipartite cylinders and bipartite tori, in: Proceedings of the CSIT Conference, 2007, pp. 86–88.
[27] P.A. Petrosyan, H.H. Khachatrian, H.G. Tananyan, Interval edge-colorings of Cartesian products of graphs I, Discuss. Math. Graph Theory 33 (3) (2013) 613–632.
[28] P.A. Petrosyan, H.H. Khachatrian, L.E. Yepremyan, H.G. Tananyan, Interval edge-colorings of graph products, in: Proceedings of the CSIT Conference, 2011, pp. 89–92.
[29] T. Pisanski, J. Shawe-Taylor, B. Mohar, 1-Factorization of the composition of regular graphs, Publ. Inst. Math. (Beograd) 33 (47) (1983) 193–196.
[30] G. Sabidussi, Graph multiplication, Math. Z. 72 (1960) 446–457.
[31] S.V. Sevast'janov, Interval colorability of the edges of a bipartite graph, Metody Diskret. Anal. 50 (1990) 61–72. (in Russian).
[32] V.G. Vizing, The Cartesian product of graphs, Vychislitelnye Sistemy 9 (1963) 30–43. (in Russian).
[33] V.G. Vizing, Some unsolved problems in graph theory, Uspekhi Mat. Nauk 23 (1968) 117–134. (in Russian).
[34] D.B. West, Introduction to Graph Theory, Prentice-Hall, New Jersey, 2001.
[35] M.K. Zhou, Decomposition of some product graphs into 1-factors and Hamiltonian cycles, Ars Combin. 28 (1989) 258–268.