Discrete Applied Mathematics 165 (2014) 263–269

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On sum edge-coloring of regular, bipartite and split graphs

P.A. Petrosyan [a,b,*], R.R. Kamalian [a,c]

[a] Institute for Informatics and Automation Problems, National Academy of Sciences, 0014, Armenia
[b] Department of Informatics and Applied Mathematics, Yerevan State University, 0025, Armenia
[c] Department of Applied Mathematics and Informatics, Russian-Armenian State University, 0051, Armenia

ARTICLE INFO

Article history:
Received 30 November 2011
Received in revised form 16 August 2013
Accepted 27 September 2013
Available online 21 October 2013

Keywords:
Edge-coloring
Sum edge-coloring
Regular graph
Bipartite graph
Split graph

ABSTRACT

An edge-coloring of a graph G with natural numbers is called a sum edge-coloring if the colors of edges incident to any vertex of G are distinct and the sum of the colors of the edges of G is minimum. The edge-chromatic sum of a graph G is the sum of the colors of edges in a sum edge-coloring of G. It is known that the problem of finding the edge-chromatic sum of an r-regular ($r \geq 3$) graph is NP-complete. In this paper we give a polynomial time $\left(1+\frac{2r}{(r+1)^2}\right)$-approximation algorithm for the edge-chromatic sum problem on r-regular graphs for $r \geq 3$. Also, it is known that the problem of finding the edge-chromatic sum of bipartite graphs with maximum degree 3 is NP-complete. We show that the problem remains NP-complete even for some restricted class of bipartite graphs with maximum degree 3. Finally, we give upper bounds for the edge-chromatic sum of some split graphs.

1. Introduction

We consider finite undirected graphs that do not contain loops or multiple edges. Let $V(G)$ and $E(G)$ denote sets of vertices and edges of G, respectively. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S, that is, $V(G[S]) = S$ and $E(G[S])$ consists of those edges of $E(G)$ for which both ends are in S. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of G by $\Delta(G)$, the chromatic number of G by $\chi(G)$, and the chromatic index of G by $\chi'(G)$. The terms and concepts that we do not define can be found in [2,26].

A proper vertex-coloring of a graph G is a mapping $\alpha : V(G) \to \mathbf{N}$ such that $\alpha(u) \neq \alpha(v)$ for every $uv \in E(G)$. If α is a proper vertex-coloring of a graph G, then $\Sigma(G, \alpha)$ denotes the sum of the colors of the vertices of G. For a graph G, define the vertex-chromatic sum $\Sigma(G)$ as follows: $\Sigma(G) = \min_\alpha \Sigma(G, \alpha)$, where minimum is taken among all possible proper vertex-colorings of G. If α is a proper vertex-coloring of a graph G and $\Sigma(G) = \Sigma(G, \alpha)$, then α is called a sum vertex-coloring. The strength of a graph G ($s(G)$) is the minimum number of colors needed for a sum vertex-coloring of G. The concept of sum vertex-coloring and vertex-chromatic sum was introduced by Kubicka [16] and Supowit [22]. In [18], Kubicka and Schwenk showed that the problem of finding the vertex-chromatic sum is NP-complete in general and polynomial time solvable for trees. Jansen [12] gave a dynamic programming algorithm for partial k-trees. In papers [5,6,9,13,17], some approximation algorithms were given for various classes of graphs. For the strength of graphs, Brook's-type theorem was proved in [11]. On the other hand, there are graphs with $s(G) > \chi(G)$ [8]. Some bounds for the vertex-chromatic sum of a graph were given in [23].

Similar to the sum vertex-coloring and vertex-chromatic sum of graphs, in [5,10,11], sum edge-coloring and edge-chromatic sum of graphs were introduced. A proper edge-coloring of a graph G is a mapping $\alpha : E(G) \to \mathbf{N}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e, e' \in E(G)$. If α is a proper edge-coloring of a graph G, then $\Sigma'(G, \alpha)$ denotes

* Corresponding author at: Institute for Informatics and Automation Problems, National Academy of Sciences, 0014, Armenia. Tel.: +374 93 27 29 32; fax: +374 10 55 46 41.
E-mail addresses: pet_petros@ipia.sci.am, pet_petros@ysu.am, pet_petros@yahoo.com (P.A. Petrosyan), rrkamalian@yahoo.com (R.R. Kamalian).

0166-218X/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.dam.2013.09.025

the sum of the colors of the edges of G. For a graph G, define the edge-chromatic sum $\Sigma'(G)$ as follows: $\Sigma'(G) = \min_\alpha \Sigma'(G,\alpha)$, where minimum is taken among all possible proper edge-colorings of G. If α is a proper edge-coloring of a graph G and $\Sigma'(G) = \Sigma'(G,\alpha)$, then α is called a sum edge-coloring. The edge-strength of a graph G ($s'(G)$) is the minimum number of colors needed for a sum edge-coloring of G. For the edge-strength of graphs, Vizing's-type theorem was proved in [11]. In [5], Bar-Noy et al. proved that the problem of finding the edge-chromatic sum is NP-hard for multigraphs. Later, in [10], it was shown that the problem is NP-complete for bipartite graphs with maximum degree 3. Also, in [10], the authors proved that the problem can be solved in polynomial time for trees and that $s'(G) = \chi'(G)$ for bipartite graphs. In [20], Salavatipour proved that the problem of determining the edge-chromatic sum and the problem of determining the edge-strength are both NP-complete for r-regular graphs with $r \geq 3$. Also he proved that $s'(G) = \chi'(G)$ for regular graphs. On the other hand, there are graphs with $\chi'(G) = \Delta(G)$ and $s'(G) = \Delta(G) + 1$ [11]. Recently, Cardinal et al. [7] determined the edge-strength of the multicycles.

In the present paper we give a polynomial time $\frac{11}{8}$-approximation algorithm for the edge-chromatic sum problem of r-regular graphs for $r \geq 3$. Next, we show that the problem of finding the edge-chromatic sum remains NP-complete even for some restricted class of bipartite graphs with maximum degree 3. Finally, we give upper bounds for the edge-chromatic sum of some split graphs.

2. Definitions and preliminary results

A proper t-coloring is a proper edge-coloring which makes use of t different colors. If α is a proper t-coloring of G and $v \in V(G)$, then $S(v,\alpha)$ denotes the set of colors appearing on edges incident to v. Let G be a graph and $R \subseteq V(G)$. A proper t-coloring of a graph G is called an R-sequential t-coloring [1,3] if the edges incident to each vertex $v \in R$ are colored by the colors $1,\ldots,d_G(v)$. For positive integers a and b, we denote by $[a,b]$, the set of all positive integers c with $a \leq c \leq b$. For a positive integer n, let K_n denote the complete graph on n vertices.

We will use the following four results.

Theorem 1 *([15]). If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.*

Theorem 2 *([24]). For every graph G,*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Theorem 3 *([25]). For the complete graph K_n with $n \geq 2$,*

$$\chi'(K_n) = \begin{cases} n-1, & \text{if } n \text{ is even},\\ n, & \text{if } n \text{ is odd}. \end{cases}$$

Theorem 4 *([10,11]). If G is a bipartite or a regular graph, then $s'(G) = \chi'(G)$.*

We also need one result on the edge-chromatic sum of complete graphs with shifted colors. First we give a definition of the shifted edge-chromatic sum. If α is a proper t-coloring of a graph G with colors $[p, p+t-1]$, then $\Sigma'_{\geq p}(G,\alpha)$ denotes the sum of the colors of the edges of G. For a graph G and $p \in \mathbf{N}$, define the shifted edge-chromatic sum $\Sigma'_{\geq p}(G)$ as follows: $\Sigma'_{\geq p}(G) = \min_\alpha \Sigma'_{\geq p}(G,\alpha)$, where minimum is taken among all possible proper edge-colorings of G with colors $p, p+1, \ldots$. The theorem we are going to prove will be used in Section 5.

Theorem 5. *For any $n, p \in \mathbf{N}$, we have*

$$\Sigma'_{\geq p}(K_n) = \begin{cases} \dfrac{n(n-1)(2p+n-1)}{4}, & \text{if } n \text{ is odd},\\[2mm] \dfrac{n(n-1)(2p+n-2)}{4}, & \text{if } n \text{ is even}. \end{cases}$$

Proof. Since for any r-regular graph G with n vertices, $\Sigma'(G) = \frac{nr(r+1)}{4}$ if and only if $\chi'(G) = r$ and, by Theorems 3 and 4, we obtain $\Sigma'_{\geq p}(K_n) = \frac{n(p+p+1+\cdots+p+n-2)}{2} = \frac{n(n-1)(2p+n-2)}{4}$ if n is even.

Now let n be an odd number and $n \geq 3$. In this case by Theorems 3 and 4, we have $s'(K_n) = \chi'(K_n) = n$. It is easy to see that in any proper n-coloring of K_n the missing colors at n vertices are all distinct. Hence,

$$\Sigma'_{\geq p}(K_n) = \frac{\frac{n^2(2p+n-1)}{2} - \frac{n(2p+n-1)}{2}}{2} = \frac{n(n-1)(2p+n-1)}{4}. \quad \square$$

Corollary 6. *For any $n \in \mathbf{N}$, we have*

$$\Sigma'(K_n) = \begin{cases} \dfrac{n(n^2-1)}{4}, & \text{if } n \text{ is odd},\\[2mm] \dfrac{(n-1)n^2}{4}, & \text{if } n \text{ is even}. \end{cases}$$

3. Edge-chromatic sums of regular graphs

In this section we consider the problem of finding the edge-chromatic sum of regular graphs. It is easy to show that the edge-chromatic sum problem of graphs G with $\Delta(G) \leq 2$ can be solved in polynomial time. On the other hand, in [19], it was proved that the problem of finding the edge-chromatic sum of an r-regular ($r \geq 3$) graph is NP-complete. Clearly, $\Sigma'(G) \geq \frac{nr(r+1)}{4}$ for any r-regular graph G with n vertices, since the sum of colors appearing on the edges incident to any vertex is at least $\frac{r(r+1)}{2}$. Moreover, it is easy to see that $\Sigma'(G) = \frac{nr(r+1)}{4}$ if and only if $\chi'(G) = r$ for any r-regular graph G with n vertices.

First we give a result on R-sequential colorings of regular graphs and then we use this result for constructing an approximation algorithm.

Theorem 7. *If G is an r-regular graph with n vertices, then G has an R-sequential $(r+1)$-coloring with $|R| \geq \left\lceil \frac{n}{r+1} \right\rceil$.*

Proof. By Theorem 2, there exists a proper $(r+1)$-coloring α of the graph G with colors $1, 2, \ldots, r+1$. For $i = 1, 2, \ldots, r+1$, define the set $V_\alpha(i)$ as follows:

$$V_\alpha(i) = \{v \in V(G) : i \notin S(v, \alpha)\}.$$

Clearly, for any i', i'', $1 \leq i' < i'' \leq r+1$, we have

$$V_\alpha(i') \cap V_\alpha(i'') = \emptyset \quad \text{and} \quad \bigcup_{i=1}^{r+1} V_\alpha(i) = V(G).$$

Hence,

$$n = |V(G)| = \left| \bigcup_{i=1}^{r+1} V_\alpha(i) \right| = \sum_{i=1}^{r+1} |V_\alpha(i)|.$$

This implies that there exists i_0, $1 \leq i_0 \leq r+1$, for which $|V_\alpha(i_0)| \geq \left\lceil \frac{n}{r+1} \right\rceil$. Let $R = V_\alpha(i_0)$.

If $i_0 = r+1$, then α is an R-sequential $(r+1)$-coloring of G; otherwise define an edge-coloring β as follows: for any $e \in E(G)$, let

$$\beta(e) = \begin{cases} \alpha(e), & \text{if } \alpha(e) \neq i_0, r+1, \\ i_0, & \text{if } \alpha(e) = r+1, \\ r+1, & \text{if } \alpha(e) = i_0. \end{cases}$$

It is easy to see that β is an R-sequential $(r+1)$-coloring of G with $|R| \geq \left\lceil \frac{n}{r+1} \right\rceil$. $\square$

Corollary 8. *If G is a cubic graph with n vertices, then G has an R-sequential 4-coloring with $|R| \geq \left\lceil \frac{n}{4} \right\rceil$.*

Note that if n is odd, then the lower bound in Theorem 7 cannot be improved, since the complete graph K_n has an R-sequential n-coloring with $|R| = 1$.

In [5], it was shown that there exists a 2-approximation algorithm for the edge-chromatic sum problem on general graphs. Now we show that there exists a $\left(1 + \frac{2r}{(r+1)^2}\right)$-approximation algorithm for the edge-chromatic sum problem on r-regular graphs for $r \geq 3$. Note that $1 + \frac{2r}{(r+1)^2}$ decreases for increasing r and $\frac{11}{8}$ is its maximum value achieved for $r = 3$. Thus, we show that there is an $\frac{11}{8}$-approximation algorithm for the edge-chromatic sum problem on regular graphs.

Theorem 9. *For any $r \geq 3$, there is a polynomial time $\left(1 + \frac{2r}{(r+1)^2}\right)$-approximation algorithm for the edge-chromatic sum problem on r-regular graphs.*

Proof. Let G be an r-regular graph with n vertices and m edges. Now we describe a polynomial time algorithm A for constructing a special proper $(r+1)$-coloring of G. First we construct a proper $(r+1)$-coloring α of G in $O(mn)$ time [21]. Next we recolor some edges as it is described in the proof of Theorem 7 to obtain an R-sequential $(r+1)$-coloring β of G with $|R| \geq \left\lceil \frac{n}{r+1} \right\rceil$. Clearly, we can do it in $O(m)$ time. Now, taking into account that the sum of colors appearing on the edges incident to any vertex is at most $\frac{r(r+3)}{2}$, we have

$$\Sigma'_A(G) = \Sigma'(G, \beta) \leq \frac{\frac{r(r+1)}{2}\left\lceil \frac{n}{r+1} \right\rceil + \left(n - \left\lceil \frac{n}{r+1} \right\rceil\right)\frac{r(r+3)}{2}}{2} \leq \frac{\frac{r(r+1)}{2}\frac{n}{r+1} + \left(n - \frac{n}{r+1}\right)\frac{r(r+3)}{2}}{2}$$
$$= \frac{\frac{r(r+1)}{2}\frac{n}{r+1} + \frac{nr}{r+1}\frac{r(r+3)}{2}}{2} = \frac{nr(r^2+4r+1)}{4(r+1)}.$$

On the other hand, since $\Sigma'(G) \geq \frac{nr(r+1)}{4}$, we get

$$\frac{\Sigma'_A(G)}{\Sigma'(G)} \leq \frac{nr(r^2+4r+1)}{4(r+1)} \cdot \frac{4}{nr(r+1)} = \frac{r^2+4r+1}{(r+1)^2} = 1 + \frac{2r}{(r+1)^2}.$$

This shows that there exists a $\left(1+\frac{2r}{(r+1)^2}\right)$-approximation algorithm for the edge-chromatic sum problem on r-regular graphs. Moreover, we can construct the aforementioned coloring β for a regular graph in $O(mn)$ time. $\square$

4. Edge-chromatic sums of bipartite graphs

In this section we consider the problem of finding the edge-chromatic sum of bipartite graphs. Let $G=(U\cup W,E)$ be a bipartite graph with bipartition (U,W). By $U_i\subseteq U$ and $W_i\subseteq W$ we denote sets of vertices of degree i in U and W, respectively. Define sets $V_{\geq i}\subseteq V(G)$ and $U_{\geq i}\subseteq U$ as follows: $V_{\geq i}=\{v: v\in V(G)\wedge d_G(v)\geq i\}$ and $U_{\geq i}=\{u\in V(G): u\in U\wedge d_G(u)\geq i\}$. The following was proved.

Theorem 10 ([1–4]). *If $G=(U\cup W,E)$ is a bipartite graph with $d_G(u)\geq d_G(w)$ for every $uw\in E(G)$, where $u\in U$ and $w\in W$, then G has a U-sequential $\Delta(G)$-coloring.*

By this theorem, we obtain the following corollary.

Corollary 11. *If $G=(U\cup W,E)$ is a bipartite graph with $d_G(u)\geq d_G(w)$ for every $uw\in E(G)$, where $u\in U$ and $w\in W$, then a U-sequential $\Delta(G)$-coloring of G is a sum edge-coloring of G and $\Sigma'(G)=\sum_{u\in U}\frac{d_G(u)(d_G(u)+1)}{2}$.*

In [10], it was shown that the problem of finding the edge-chromatic sum of bipartite graphs G with $\Delta(G)=3$ is NP-complete. Now we give a short proof of this fact. First we need the following:

Problem 1 ([2,3,14]).
Instance: A bipartite graph $G=(U\cup W,E)$ with $\Delta(G)=3$.
Question: Is there a U-sequential 3-coloring of G?
The following was proved.

Theorem 12 ([3,14]). *Problem 1 is NP-complete.*

Now let us consider the following:

Problem 2.
Instance: A bipartite graph $G=(U\cup W,E)$ with $\Delta(G)=3$.
Question: Is $\Sigma'(G)=\sum_{i=1}^{3} i\cdot\left|U_{\geq i}\right|$?

Theorem 13. *Problem 2 is NP-complete.*

Proof. Clearly, Problem 2 belongs to NP. For the proof of the NP-completeness, we show a reduction from Problem 1 to Problem 2. We prove that a bipartite graph $G=(U\cup W,E)$ with $\Delta(G)=3$ admits a U-sequential 3-coloring if and only if $\Sigma'(G)=\sum_{i=1}^{3} i\cdot\left|U_{\geq i}\right|$. Let $G=(U\cup W,E)$ be a bipartite graph with $\Delta(G)=3$ and α be a U-sequential 3-coloring of G. In this case the colors 1, 2, 3 appear on the edges incident to each vertex $u\in U_3$, the colors 1, 2 appear on the edges incident to each vertex $u\in U_2$ and the color 1 appears on the pendant edges incident to each vertex $u\in U_1$. Hence, $\Sigma'(G,\alpha)=\sum_{i=1}^{3} i\cdot\left|U_{\geq i}\right|$. On the other hand, clearly, $\Sigma'(G)\geq\sum_{i=1}^{3} i\cdot\left|U_{\geq i}\right|$; thus $\Sigma'(G)=\sum_{i=1}^{3} i\cdot\left|U_{\geq i}\right|$.

Now suppose that $\Sigma'(G)=\sum_{i=1}^{3} i\cdot\left|U_{\geq i}\right|$. By Theorems 1 and 4, there exists a proper 3-coloring β of a bipartite graph G with $\Delta(G)=3$. This implies that the colors 1, 2, 3 appear on the edges incident to each vertex $u\in U_3$. If the color 3 appears on the edges incident to some vertices $u\in U_2$ or the color 2 or 3 appears on the pendant edges incident to some vertices $u\in U_1$, then it is easy to see that $\Sigma'(G,\beta)>\sum_{i=1}^{3} i\cdot\left|U_{\geq i}\right|$. Hence, β is a U-sequential 3-coloring of G. $\square$

Now we prove that the problem of finding the edge-chromatic sum of bipartite graphs G with $\Delta(G)=3$ and with additional conditions is NP-complete, too. We need the following:

Problem 3 ([3,14]).
Instance: A bipartite graph $G=(U\cup W,E)$ with $\Delta(G)=3$ and $|U_i|=|W_i|$ for $i=1,2,3$.
Question: Is there a $V(G)$-sequential 3-coloring of G?
The following was proved.

Theorem 14 ([3,14]). *Problem 3 is NP-complete.*

Now let us consider the following:
Problem 4.
Instance: A bipartite graph $G=(U\cup W,E)$ with $\Delta(G)=3$ and $|U_i|=|W_i|$ for $i=1,2,3$.
Question: Is $\Sigma'(G)=\frac{1}{2}\sum_{i=1}^{3} i\cdot\left|V_{\geq i}\right|$?

Theorem 15. *Problem 4 is NP-complete.*

Proof. Clearly, Problem 4 belongs to NP. For the proof of the NP-completeness, we show a reduction from Problem 3 to Problem 4. We prove that a bipartite graph $G=(U\cup W,E)$ with $\Delta(G)=3$ and $|U_i|=|W_i|$ for $i=1,2,3$, admits a $V(G)$-sequential 3-coloring if and only if $\Sigma'(G)=\frac{1}{2}\sum_{i=1}^{3} i\cdot\left|V_{\geq i}\right|$. Let α be a $V(G)$-sequential 3-coloring of G. In this case the colors

1, 2, 3 appear on the edges incident to each vertex $v \in V(G)$ with $d_G(v) = 3$, the colors 1, 2 appear on the edges incident to each vertex $v \in V(G)$ with $d_G(v) = 2$ and the color 1 appears on the pendant edges incident to each vertex $v \in V(G)$ with $d_G(v) = 1$. Hence, $\Sigma'(G, \alpha) = \frac{1}{2}\sum_{i=1}^{3} i \cdot \left|V_{\geq i}\right|$. On the other hand, clearly, $\Sigma'(G) \geq \frac{1}{2}\sum_{i=1}^{3} i \cdot \left|V_{\geq i}\right|$; thus $\Sigma'(G) = \frac{1}{2}\sum_{i=1}^{3} i \cdot \left|V_{\geq i}\right|$.

Now suppose that $\Sigma'(G) = \frac{1}{2}\sum_{i=1}^{3} i \cdot \left|V_{\geq i}\right|$. By Theorems 1 and 4, there exists a proper 3-coloring β of a bipartite graph G with $\Delta(G) = 3$ and $|U_i| = |W_i|$ for $i = 1, 2, 3$. This implies that the colors 1, 2, 3 appear on the edges incident to each vertex $v \in V(G)$ with $d_G(v) = 3$. If the color 3 appears on the edges incident to some vertices $v \in V(G)$ with $d_G(v) = 2$ or the color 2 or 3 appears on the pendant edges incident to some vertices $v \in V(G)$ with $d_G(v) = 1$, then it is easy to see that $\Sigma'(G, \beta) > \frac{1}{2}\sum_{i=1}^{3} i \cdot \left|V_{\geq i}\right|$. Hence, β is a $V(G)$-sequential 3-coloring of G. $\square$

In [19], it was proved that the problem of finding the edge-chromatic sum of bipartite graphs G with $\Delta(G) = 3$ remains NP-hard even for planar bipartite graphs.

5. Edge-chromatic sums of split graphs

In this section we consider the problem of finding the edge-chromatic sum of split graphs. A split graph is a graph whose vertices can be partitioned into a clique C and an independent set I. Let $G = (C \cup I, E)$ be a split graph, where $C = \{u_1, u_2, \ldots, u_n\}$ is a clique and $I = \{v_1, v_2, \ldots, v_m\}$ is an independent set. Define a number Δ_I as follows: $\Delta_I = \max_{1\leq j\leq m} d_G(v_j)$. Define subgraphs H and H' of a graph G as follows:

$$H = (C \cup I, E(G) \setminus E(G[C])) \quad \text{and} \quad H' = G[C].$$

Clearly, H is a bipartite graph with bipartition (C, I), and $d_H(u_i) = d_G(u_i) - n + 1$ for $i = 1, 2, \ldots, n$, $d_H(v_j) = d_G(v_j)$ for $j = 1, 2, \ldots, m$.

Theorem 16. *Let $G = (C \cup I, E)$ be a split graph, where $C = \{u_1, u_2, \ldots, u_n\}$ is a clique and $I = \{v_1, v_2, \ldots, v_m\}$ is an independent set. If $d_G(u_i) - d_G(v_j) \geq n - 1$ for every $u_iv_j \in E(G)$, then*

(1) *if n is even, then*

$$\Sigma'(G) \leq \min\left\{\sum_{i=1}^{n}\frac{(d_G(u_i)-n+1)\,(d_G(u_i)-n+2)}{2} + \Sigma'_{\geq\Delta(G)-n+2}(K_n),\right.$$
$$\left.\Sigma'(K_n) + \sum_{i=1}^{n}\frac{(d_G(u_i)-n+1)\,(d_G(u_i)+n)}{2}\right\},$$

(2) *if n is odd, then*

$$\Sigma'(G) \leq \min\left\{\sum_{i=1}^{n}\frac{(d_G(u_i)-n+1)\,(d_G(u_i)-n+2)}{2} + \Sigma'_{\geq\Delta(G)-n+2}(K_n),\right.$$
$$\left.\Sigma'(K_n) + \sum_{i=1}^{n}\frac{(d_G(u_i)-n+1)\,(d_G(u_i)+n+2)}{2}\right\}.$$

Proof. For the proof, we are going to construct edge-colorings that satisfy the specified conditions.

Since $d_G(u_i) - d_G(v_j) \geq n - 1$ for every $u_iv_j \in E(G)$, we have $d_H(u_i) \geq d_H(v_j)$ for each $u_iv_j \in E(H)$. By Theorem 10, there exists a C-sequential $\Delta(H)$-coloring α of the graph H and, by Corollary 11, we obtain

$$\Sigma'(H) = \Sigma'(H, \alpha) = \sum_{i=1}^{n}\frac{d_H(u_i)\,(d_H(u_i)+1)}{2}.$$

Now we consider two cases.

Case 1: n is even.

In this case, by Theorem 3, we have $\chi'(H') = n - 1$. Let β be a proper edge-coloring of a graph H' with colors $\Delta(G) - n + 2, \ldots, \Delta(G)$. By Theorem 5, we obtain

$$\Sigma'(G) \leq \Sigma'(H) + \Sigma'_{\geq\Delta(G)-n+2}(K_n).$$

On the other hand, let β' be a proper edge-coloring of a graph H' with colors $1, 2, \ldots, n - 1$. Next we define an edge-coloring γ of the graph H as follows: for every $e \in E(H)$, let $\gamma(e) = \alpha(e) + n - 1$. Thus, by Corollary 6, we obtain

$$\Sigma'(G) \leq \Sigma'(K_n) + \sum_{i=1}^{n}\frac{(d_G(u_i)-n+1)\,(d_G(u_i)+n)}{2}.$$

Case 2: n is odd.

In this case, by Theorem 3, we have $\chi'(H') = n$. Let β be a proper edge-coloring of a graph H' with colors $\Delta(G) - n + 2, \ldots, \Delta(G) + 1$. By Theorem 5, we obtain

$$\Sigma'(G) \leq \Sigma'(H) + \Sigma'_{\geq \Delta(G)-n+2}(K_n).$$

On the other hand, let β' be a proper edge-coloring of a graph H' with colors $1, 2, \ldots, n$. Next we define an edge-coloring γ of the graph H as follows: for every $e \in E(H)$, let $\gamma(e) = \alpha(e) + n$. Thus, by Corollary 6, we obtain

$$\Sigma'(G) \leq \Sigma'(K_n) + \sum_{i=1}^{n} \frac{(d_G(u_i) - n + 1)\,(d_G(u_i) + n + 2)}{2}. \quad \square$$

Theorem 17. *Let $G = (C \cup I, E)$ be a split graph, where $C = \{u_1, u_2, \ldots, u_n\}$ is a clique and $I = \{v_1, v_2, \ldots, v_m\}$ is an independent set. If $d_G(u_i) - d_G(v_j) \leq n - 1$ for every $u_i v_j \in E(G)$, then*

(1) *if n is even, then*

$$\Sigma'(G) \leq \min\left\{\sum_{j=1}^{m} \frac{d_G(v_j)\left(d_G(v_j)+1\right)}{2} + \Sigma'_{\geq \Delta_I+1}(K_n),\ \Sigma'(K_n) + \sum_{j=1}^{m} \frac{d_G(v_j)\left(d_G(v_j)+2n-1\right)}{2}\right\},$$

(2) *if n is odd, then*

$$\Sigma'(G) \leq \min\left\{\sum_{j=1}^{m} \frac{d_G(v_j)\left(d_G(v_j)+1\right)}{2} + \Sigma'_{\geq \Delta_I+1}(K_n),\ \Sigma'(K_n) + \sum_{j=1}^{m} \frac{d_G(v_j)\left(d_G(v_j)+2n+1\right)}{2}\right\}.$$

Proof. For the proof, we are going to construct edge-colorings that satisfy the specified conditions.

Since $d_G(u_i) - d_G(v_j) \leq n - 1$ for every $u_i v_j \in E(G)$, we have $d_H(u_i) \leq d_H(v_j)$ for each $u_i v_j \in E(H)$. By Theorem 10, there exists an I-sequential Δ_I-coloring α of the graph H and, by Corollary 11, we obtain

$$\Sigma'(H) = \Sigma'(H, \alpha) = \sum_{j=1}^{m} \frac{d_H(v_j)\left(d_H(v_j)+1\right)}{2} = \sum_{j=1}^{m} \frac{d_G(v_j)\left(d_G(v_j)+1\right)}{2}.$$

Now we consider two cases.

Case 1: n is even.

In this case, by Theorem 3, we have $\chi'(H') = n - 1$. Let β be a proper edge-coloring of a graph H' with colors $\Delta_I + 1, \ldots, \Delta_I + n - 1$. By Theorem 5, we obtain

$$\Sigma'(G) \leq \Sigma'(H) + \Sigma'_{\geq \Delta_I+1}(K_n).$$

On the other hand, let β' be a proper edge-coloring of a graph H' with colors $1, 2, \ldots, n - 1$. Next we define an edge-coloring γ of the graph H as follows: for every $e \in E(H)$, let $\gamma(e) = \alpha(e) + n - 1$. Thus, by Corollary 6, we obtain

$$\Sigma'(G) \leq \Sigma'(K_n) + \sum_{j=1}^{m} \frac{d_G(v_j)\left(d_G(v_j)+2n-1\right)}{2}.$$

Case 2: n is odd.

In this case, by Theorem 3, we have $\chi'(H') = n$. Let β be a proper edge-coloring of a graph H' with colors $\Delta_I + 1, \ldots, \Delta_I + n$. By Theorem 5, we obtain

$$\Sigma'(G) \leq \Sigma'(H) + \Sigma'_{\geq \Delta_I+1}(K_n).$$

On the other hand, let β' be a proper edge-coloring of a graph H' with colors $1, 2, \ldots, n$. Next we define an edge-coloring γ of the graph H as follows: for every $e \in E(H)$, let $\gamma(e) = \alpha(e) + n$. Thus, by Corollary 6, we obtain

$$\Sigma'(G) \leq \Sigma'(K_n) + \sum_{j=1}^{m} \frac{d_G(v_j)\left(d_G(v_j)+2n+1\right)}{2}. \quad \square$$

Acknowledgments

We would like to thank both referees for many useful suggestions.

References

[1] A.S. Asratian, Investigation of some mathematical model of scheduling theory, Doctoral Thesis, Moscow, 1980.
[2] A.S. Asratian, T.M.J. Denley, R. Haggkvist, Bipartite Graphs and Their Applications, Cambridge University Press, Cambridge, 1998.
[3] A.S. Asratian, R.R. Kamalian, Interval colorings of edges of a multigraph, Appl. Math. 5 (1987) 25–34 (in Russian).
[4] A.S. Asratian, R.R. Kamalian, Investigation on interval edge-colorings of graphs, J. Combin. Theory Ser. B 62 (1994) 34–43.

[5] A. Bar-Noy, M. Bellare, M.M. Halldorsson, H. Shachnai, T. Tamir, On chromatic sums and distributed resource allocation, Inform. and Comput. 140 (1998) 183–202.
[6] A. Bar-Noy, G. Kortsarz, The minimum color sum of bipartite graphs, J. Algorithms 28 (1998) 339–365.
[7] J. Cardinal, V. Ravelomanana, M. Valencia-Pabon, Minimum sum edge colorings of multicycles, Discrete Appl. Math. 158 (2010) 1216–1223.
[8] P. Erdős, E. Kubicka, A. Schwenk, Graphs that require many colors to achieve their chromatic sum, Congr. Numer. 71 (1990) 17–28.
[9] K. Giaro, R. Janczewski, M. Kubale, M. Malafiejski, A 27/26-approximation algorithm for the chromatic sum coloring of bipartite graphs, Lecture Notes in Comput. Sci. 2462 (2002) 135–145.
[10] K. Giaro, M. Kubale, Edge-chromatic sum of trees and bounded cyclicity graphs, Inform. Process. Lett. 75 (2000) 65–69.
[11] H. Hajiabolhassan, M.L. Mehrabadi, R. Tusserkani, Minimal coloring and strength of graphs, Discrete Math. 215 (2000) 265–270.
[12] K. Jansen, The optimum cost chromatic partition Problem, in: Proc. of The 3rd Italian Conference on Algorithms and Complexity, CIAC'97, in: Lecture Notes in Computer Science, vol. 1203, 1997, pp. 25–36.
[13] K. Jansen, Approximation results for the optimum cost chromatic partition problem, J. Algorithms 34 (2000) 54–89.
[14] R.R. Kamalian, Interval edge-colorings of graphs, Doctoral Thesis, Novosibirsk, 1990.
[15] D. König, Über graphen und ihre anwendung auf determinantentheorie und mengenlehre, Math. Ann. 77 (1916) 453–465.
[16] E. Kubicka, The chromatic sum of a graph, PhD Thesis, Western Michigan University, 1989.
[17] E. Kubicka, G. Kubicki, D. Kountanis, Approximation algorithms for the chromatic sum, in: Proc. of The First Great Lakes Computer Science Conference, in: Lecture Notes in Computer Science, vol. 507, 1989, pp. 15–21.
[18] E. Kubicka, A.J. Schwenk, An introduction to chromatic sums, in: Proc. of The 17th Annual ACM Computer Science Conference, ACM Press, New York, 1989, pp. 39–45.
[19] D. Marx, Complexity results for minimum sum edge coloring, Discrete Appl. Math. 157 (2009) 1034–1045.
[20] M.R. Salavatipour, On sum coloring of graphs, Discrete Appl. Math. 127 (2003) 477–488.
[21] A. Schrijver, Combinatorial Optimization, Polyhedra and Efficiency, Springer-Verlag, Heidelberg, 2004.
[22] K.J. Supowit, Finding a maximum planar subset of nets in a channel, IEEE Trans. Comput.-Aided Design CAD 6 (1) (1987) 93–94.
[23] C. Thomassen, P. Erdős, Y. Alavi, P.J. Malde, A.J. Schwenk, Tight bounds on the chromatic sum of a connected graph, J. Graph Theory 13 (1989) 353–357.
[24] V.G. Vizing, On an estimate of the chromatic class of a p-graph, Disc. Analiz. 3 (1964) 25–30 (in Russian).
[25] V.G. Vizing, The chromatic class of a multigraph, Kibernetika 3 (1965) 29–39 (in Russian).
[26] D.B. West, Introduction to Graph Theory, Prentice-Hall, New Jersey, 2001.