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Interval edge-colorings of complete graphs

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ABSTRACT

An edge-coloring of a graph G with colors $1, 2, \ldots, t$ is an *interval t-coloring* if all colors are used, and the colors of edges incident to each vertex of G are distinct and form an interval of integers. A graph G is *interval colorable* if it has an interval t-coloring for some positive integer t. For an interval colorable graph G, $W(G)$ denotes the greatest value of t for which G has an interval t-coloring. It is known that the complete graph is interval colorable if and only if the number of its vertices is even. However, the exact value of $W(K_{2n})$ is known only for $n \leq 4$. The second author showed that if $n = p2^q$, where p is odd and q is nonnegative, then $W(K_{2n}) \geq 4n - 2 - p - q$. Later, he conjectured that if $n \in \mathbb{N}$, then $W(K_{2n}) = 4n - 2 - \lfloor \log_2 n \rfloor - \|n_2\|$, where $\|n_2\|$ is the number of 1's in the binary representation of n.

In this paper we introduce a new technique to construct interval colorings of complete graphs based on their 1-factorizations, which is used to disprove the conjecture, improve lower and upper bounds on $W(K_{2n})$ and determine its exact values for $n \leq 12$.

1. Introduction

All graphs in this paper are finite, undirected, have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S, that is, $V(G[S]) = S$ and $E(G[S])$ consists of those edges of $E(G)$ for which both ends are in S. For a graph G, $\Delta(G)$ denotes the maximum degree of vertices in G. A graph G is *r-regular* if all its vertices have degree r. The set of edges M is called a *matching* if no two edges from M are adjacent. A vertex v is *covered* by the matching M if it is incident to one of the edges of M. A matching M is a *perfect matching* if it covers all the vertices of the graph G. The set of perfect matchings $\mathfrak{F} = \{F_1, F_2, \ldots, F_n\}$ is a 1-*factorization* of G if every edge of G belongs to exactly one of the perfect matchings in $\mathfrak{F}$. The set of integers $\{a, a+1, \ldots, b\}$, $a \leq b$, is denoted by $[a, b]$. The terms, notations and concepts that we do not define can be found in [14].

A *proper edge-coloring* of graph G is a coloring of the edges of G such that no two adjacent edges receive the same color. The *chromatic index* $\chi'(G)$ of a graph G is the minimum number of colors used in a proper edge-coloring of G. If α is a proper edge-coloring of G and $v \in V(G)$, then the *spectrum* of a vertex v, denoted by $S(v, \alpha)$, is the set of colors of edges incident to v. By $\underline{S}(v, \alpha)$ and $\overline{S}(v, \alpha)$ we denote the smallest and largest colors of the spectrum, respectively. If α is a proper edge-coloring

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of G and H is a subgraph of G, then we can define a union and intersection of spectrums of the vertices of H:

$$S_{\cap}(H,\alpha) = \bigcap_{v\in V(H)} S(v,\alpha)$$
$$S_{\cup}(H,\alpha) = \bigcup_{v\in V(H)} S(v,\alpha).$$

A proper edge-coloring of a graph G with colors $1, 2, \dots, t$ is an *interval t-coloring* if all colors are used, and for any vertex v of G, the set $S(v,\alpha)$ is an interval of consecutive integers. A graph G is *interval colorable* if it has an interval t-coloring for some positive integer t. The set of interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the least and the greatest values of t for which G has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively.

The concept of interval edge-coloring was introduced by Asratian and Kamalian [1]. In [1,2], they proved that if G is interval colorable, then $\chi'(G) = \Delta(G)$. For regular graphs the converse is also true. Moreover, if $G \in \mathfrak{N}$ is regular, then $w(G) = \Delta(G)$ and G has an interval t-coloring for every t, $w(G) \le t \le W(G)$. For a complete graph K_m, Vizing [13] proved that $\chi'(K_m) = m - 1$ if m is even and $\chi'(K_m) = m$ if m is odd. These results imply that the complete graph is interval colorable if and only if the number of vertices is even. Moreover, $w(K_{2n}) = 2n - 1$, for any $n \in \mathbb{N}$. On the other hand, the problem of determining the exact value of $W(K_{2n})$ is open since 1990.

In [6] Kamalian proved the following upper bound on $W(G)$:

Theorem 1. *If G is a connected graph with at least two vertices and $G \in \mathfrak{N}$, then $W(G) \le 2|V(G)| - 3$.*

This upper bound was improved by Giaro, Kubale, Malafiejski in [4]:

Theorem 2. *If G is a connected graph with at least three vertices and $G \in \mathfrak{N}$, then $W(G) \le 2|V(G)| - 4$.*

Improved upper bounds on $W(G)$ are known for several classes of graphs, including triangle-free graphs [1,2], planar graphs [3] and r-regular graphs with at least $2r + 2$ vertices [7]. The exact value of the parameter $W(G)$ is known for even cycles, trees [5], complete bipartite graphs [5], Möbius ladders [10] and n-dimensional cubes [11,12]. This paper is focused on investigation of $W(K_{2n})$.

The first lower bound on $W(K_{2n})$ was obtained by Kamalian in [6]:

Theorem 3. *For any $n \in \mathbb{N}$, $W(K_{2n}) \ge 2n - 1 + \lfloor \log_2(2n-1) \rfloor$.*

This bound was improved by the second author in [11]:

Theorem 4. *For any $n \in \mathbb{N}$, $W(K_{2n}) \ge 3n - 2$.*

In the same paper he also proved the following statement:

Theorem 5. *For any $n \in \mathbb{N}$, $W(K_{4n}) \ge 4n - 1 + W(K_{2n})$.*

By combining these two results he obtained an even better lower bound on $W(K_{2n})$:

Theorem 6. *If $n = p2^q$, where p is odd, $q \in \mathbb{Z}_+$, then $W(K_{2n}) \ge 4n - 2 - p - q$.*

In that paper the second author also posed the following conjecture:

Conjecture 1. *If $n = p2^q$, where p is odd, $q \in \mathbb{Z}_+$, then $W(K_{2n}) = 4n - 2 - p - q$.*

He verified this conjecture for $n \le 4$, but the first author disproved it by constructing an interval 14-coloring of K_{10} in [8]. In "Cycles and Colorings 2012" workshop the second author presented another conjecture on $W(K_{2n})$:

Conjecture 2. *If $n \in \mathbb{N}$, then $W(K_{2n}) = 4n - 2 - \lfloor \log_2 n \rfloor - \|n_2\|$, where $\|n_2\|$ is the number of 1's in the binary representation of n.*

In Section 2 we show that the problem of constructing an interval coloring of a complete graph K_{2n} is equivalent to finding a special 1-factorization of the same graph. In Section 3 we use this equivalence to improve the lower bounds of Theorems 4 and 5, and disprove Conjecture 2. Section 4 improves the upper bound of Theorem 2 for complete graphs. In Section 5 we determine the exact values of $W(K_{2n})$ for $n \le 12$ and improve Theorem 6.

2. From interval colorings to 1-factorizations

Let the vertex set of a complete graph K_{2n} be $V(K_{2n}) = \{u_i, v_i \mid i = 1, 2, \dots, n\}$. For any fixed ordering of the vertices $\mathbf{v} = (u_1, v_1, u_2, v_2, \dots, u_n, v_n)$ we denote by $H_{\mathbf{v}}^{[i,j]}$, $i \le j$, the subgraph of K_{2n} induced by the vertices $u_i, v_i, u_{i+1}, v_{i+1}, \dots, u_j, v_j$.

Let $\mathfrak{F} = \{F_1, F_2, \dots, F_{2n-1}\}$ be a 1-factorization of K_{2n}. For every $F \in \mathfrak{F}$ we define its *left and right parts* with respect to the ordering of vertices $\mathbf{v}$:

$$l_{\mathbf{v}}^{i}(F) = F \cap E\left(H_{\mathbf{v}}^{[1,i]}\right)$$
$$r_{\mathbf{v}}^{i}(F) = F \cap E\left(H_{\mathbf{v}}^{[i+1,n]}\right).$$

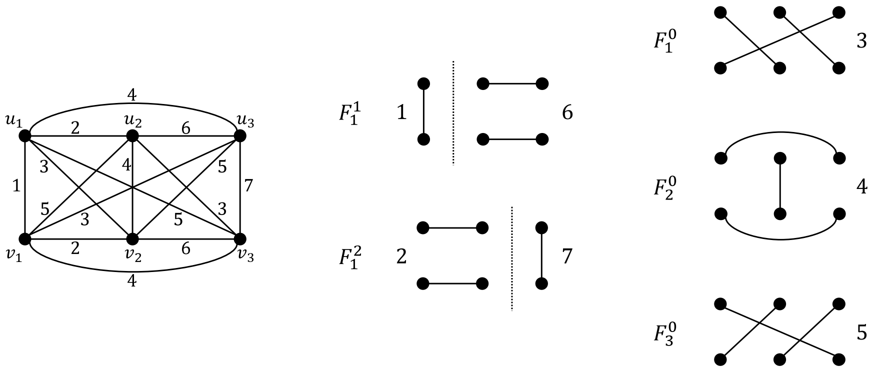

Fig. 1. Interval 7-coloring of K_6 and the corresponding 1-factorization $\mathfrak{F} = \{F_1^1, F_1^2, F_1^0, F_2^0, F_3^0\}$.

If for some i, $1 \le i \le n-1$, $F = l_{\mathbf{v}}^i(F) \cup r_{\mathbf{v}}^i(F)$ then F is called an *i-splitted* perfect matching with respect to the ordering $\mathbf{v}$. In other words the edges of F do not cross the vertical line between the ith and $(i+1)$th pairs of vertices (F_1^1 and F_1^2 on Fig. 1).

Let α be any interval edge-coloring of K_{2n}. By renaming the vertices we can achieve the following inequalities:

$$\underline{S}(u_i, \alpha) \le \underline{S}(v_i, \alpha) \le \underline{S}(u_{i+1}, \alpha) \le \underline{S}(v_{i+1}, \alpha), \quad i = 1, 2, \ldots, n-1.$$

So every coloring α implies a special ordering of vertices $\mathbf{v}_\alpha = (u_1, v_1, u_2, v_2, \ldots, u_n, v_n)$ for which these inequalities are satisfied.

Now we fix the ordering $\mathbf{v}_\alpha$ and investigate some properties of the coloring α. First we show that the spectrums of the vertices u_i and v_i are the same.

Remark 1. For every α interval edge-coloring of K_{2n}, $S_\cap(K_{2n}, \alpha) \neq \emptyset$. Otherwise it would contradict the upper bound in Theorem 1.

Lemma 1. *If* $1 \le i \le n$, *then* $\underline{S}(u_i, \alpha) = \underline{S}(v_i, \alpha)$.

Proof. Remark 1 implies that if $\underline{S}(v_i, \alpha) - \underline{S}(u_i, \alpha) > 0$, then the edges colored by $\underline{S}(u_i, \alpha)$ form a perfect matching in the subgraph $K_{2n}[\{u_1, v_1, u_2, v_2, \ldots, u_i\}]$, which is impossible, as it has odd number of vertices. $\square$

For the coloring α we define its *shift vector* in the following way:

$$\mathrm{sh}(\alpha) = (b_1, b_2, \ldots, b_{n-1})$$

where $b_i = \underline{S}(u_{i+1}, \alpha) - \underline{S}(u_i, \alpha), \quad i = 1, 2, \ldots, n-1.$

By B_i we denote the partial sums: $B_0 = 0$ and $B_i = \sum_{j=1}^{i} b_j$, $i = 1, 2, \ldots, n-1$.
The *total shift* of the coloring α is defined as follows:

$$|\mathrm{sh}(\alpha)| = B_{n-1} = \sum_{i=1}^{n-1} b_i.$$

Remark 2. If α is an interval t-coloring of K_{2n} and $\mathrm{sh}(\alpha) = (b_1, b_2, \ldots, b_{n-1})$, then $t = 2n - 1 + |\mathrm{sh}(\alpha)|$.

Remark 3. For every α interval edge-coloring of K_{2n}, the colors that appear in all vertices are $S_\cap(K_{2n}, \alpha) = \left[\underline{S}(u_n, \alpha), \overline{S}(u_1, \alpha)\right] = [|\mathrm{sh}(\alpha)| + 1, 2n-1] = \{|\mathrm{sh}(\alpha)| + j \mid j = 1, 2, \ldots, 2n-1-|\mathrm{sh}(\alpha)|\}$

For every $i = 1, 2, \ldots, n-1$, we define the following two sets of colors:

$$L_{\mathbf{v}_\alpha}^i(\alpha) = \begin{cases} [\underline{S}(u_i, \alpha), \underline{S}(u_{i+1}, \alpha) - 1] = \{B_{i-1} + j \mid j = 1, 2, \ldots, b_i\}, & \text{if } b_i > 0, \\ \emptyset, & \text{if } b_i = 0, \end{cases}$$

$$R_{\mathbf{v}_\alpha}^i(\alpha) = \begin{cases} [\overline{S}(u_i, \alpha) + 1, \overline{S}(u_{i+1}, \alpha)] = \{B_{i-1} + 2n - 1 + j \mid j = 1, 2, \ldots, b_i\}, & \text{if } b_i > 0, \\ \emptyset, & \text{if } b_i = 0. \end{cases}$$

Remark 4. If α is an interval t-coloring of K_{2n} and $\mathrm{sh}(\alpha) = (b_1, b_2, \ldots, b_{n-1})$, then

$$\begin{array}{ll} L_{\mathbf{v}_\alpha}^i(\alpha) \subset S_\cap\left(H_{\mathbf{v}_\alpha}^{[1,i]}, \alpha\right) & L_{\mathbf{v}_\alpha}^i(\alpha) \cap S_\cup\left(H_{\mathbf{v}_\alpha}^{[i+1,n]}, \alpha\right) = \emptyset \\ R_{\mathbf{v}_\alpha}^i(\alpha) \cap S_\cup\left(H_{\mathbf{v}_\alpha}^{[1,i]}, \alpha\right) = \emptyset & R_{\mathbf{v}_\alpha}^i(\alpha) \subset S_\cap\left(H_{\mathbf{v}_\alpha}^{[i+1,n]}, \alpha\right). \end{array}$$

By $C_k(\alpha)$ we denote the edges colored by the color k: $C_k(\alpha) = \{e \in E(K_{2n}) \mid \alpha(e) = k\}$.

Lemma 2 (*Equivalence Lemma*). *The following two statements are equivalent:*

(a) *there exists α interval edge-coloring of K_{2n} such that* $\mathrm{sh}(\alpha) = (b_1, b_2, \ldots, b_{n-1})$,

(b) *there exist* $\mathbf{v}$ *ordering of vertices and* $\mathfrak{F} = \left\{F_j^0 \mid j = 1, 2, \ldots, 2n-1-\sum_{i=1}^{n-1} b_i\right\} \cup \bigcup_{i=1}^{n-1}\left\{F_j^i \mid j = 1, 2, \ldots, b_i\right\}$ *1-factorization of K_{2n} such that F_j^i is i-splitted with respect to the ordering* $\mathbf{v}$, $i = 1, 2, \ldots, n-1, j = 1, 2, \ldots, b_i$, $b_i \in \mathbb{Z}_+$.

Proof. Throughout the proof we will use B_i as a shorthand for $\sum_{j=1}^{i} b_j$, $i = 0, 1, \ldots, n-1$.

(a) => (b). Let α be an interval t-coloring of K_{2n} such that $\mathrm{sh}(\alpha) = (b_1, b_2, \ldots, b_{n-1})$. We choose the ordering $\mathbf{v}_\alpha$ and construct the 1-factorization $\mathfrak{F}$ of K_{2n}.

According to Remark 3, there exist $2n-1-|\mathrm{sh}(\alpha)|$ colors that appear in the spectrums of all the vertices. By definition, $|\mathrm{sh}(\alpha)| = \sum_{i=1}^{n-1} b_i$, so we take $F_j^0 = C_{|\mathrm{sh}(\alpha)|+j}(\alpha)$, for every $j = 1, 2, \ldots, 2n-1-|\mathrm{sh}(\alpha)|$.

For every $i = 1, 2, \ldots, n-1$, Remark 4 implies there exist $|L^i_{\mathbf{v}_\alpha}(\alpha)| = b_i$ distinct colors that appear only in the spectrums of the first i pairs of vertices and another $|R^i_{\mathbf{v}_\alpha}(\alpha)| = b_i$ distinct colors that appear only in the spectrums of the remaining $2n-2i$ vertices. We take $F_j^i = C_{B_{i-1}+j}(\alpha) \cup C_{B_{i-1}+2n-1+j}(\alpha)$, for every $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, b_i$. Note that the edges colored by the colors from $L^i_{\mathbf{v}_\alpha}(\alpha) \cup R^i_{\mathbf{v}_\alpha}(\alpha)$ do not cross the vertical line between the ith and $(i+1)$th pairs of vertices (F_1^1 and F_1^2 on Fig. 1), so F_j^i is i-splitted with respect to the ordering $\mathbf{v}_\alpha$ for all permitted j.

(b) => (a). Suppose $\mathfrak{F} = \left\{F_j^0 \mid j = 1, 2, \ldots, 2n-1-|\mathrm{sh}(\alpha)|\right\} \cup \bigcup_{i=1}^{n-1}\left\{F_j^i \mid j = 1, 2, \ldots, b_i\right\}$ is a 1-factorization of K_{2n} with the property that F_j^i is i-splitted perfect matching with respect to the ordering $\mathbf{v} = (u_1, v_1, u_2, v_2, \ldots, u_n, v_n)$, $i = 1, 2, \ldots, n-1, j = 1, 2, \ldots, b_i$. We construct α interval edge-coloring of K_{2n} in the following way:

$$\begin{array}{ll}
\alpha(e) = B_{i-1} + j & \text{if } e \in l^i_{\mathbf{v}}(F^i_j)\ i = 1, 2, \ldots, n-1, j = 1, 2, \ldots, b_i \\
\alpha(e) = B_{n-1} + j & \text{if } e \in F^0_j\ j = 1, 2, \ldots, 2n-1-B_{n-1} \\
\alpha(e) = B_{i-1} + 2n - 1 + j & \text{if } e \in r^i_{\mathbf{v}}(F^i_j)\ i = 1, 2, \ldots, n-1, j = 1, 2, \ldots, b_i.
\end{array}$$

The fact that F_j^i is i-splitted with respect to the ordering $\mathbf{v}$ implies that every edge of K_{2n} has received a color. The vertex u_i (also v_i) is covered by all perfect matchings F_j^0, $j = 1, 2, \ldots, 2n-1-B_{n-1}$, by the left parts of the matchings $F_j^{i'}$, $i' = i, i+1, \ldots, n-1$, and by the right parts of the matchings $F_j^{i'}$, $i' = 1, 2, \ldots, i-1$, for every $j = 1, 2, \ldots, b_{i'}$. So the spectrum is:

$$\begin{aligned}
S(u_i, \alpha) = S(v_i, \alpha) &= \bigcup_{i'=i}^{n-1}\{B_{i'-1} + j \mid j = 1, 2, \ldots, b_{i'}\} \\
&\cup \{B_{n-1} + j \mid j = 1, 2, \ldots, 2n-1-B_{n-1}\} \\
&\cup \bigcup_{i'=1}^{i-1}\{B_{i'-1} + 2n - 1 + j \mid j = 1, 2, \ldots, b_{i'}\} \\
&= [B_{i-1}+1, B_{n-1}] \cup [B_{n-1}+1, 2n-1] \cup [2n, B_{i-1}+2n-1] \\
&= [B_{i-1}+1, B_{i-1}+2n-1].
\end{aligned}$$

This proves that α is an interval $(B_{n-1}+2n-1)$-coloring of K_{2n}. To complete the proof of the lemma we need to check the shift vector of the coloring α. Note that for every $i = 1, 2, \ldots, n-1$, we have $\underline{S}(u_{i+1}, \alpha) - \underline{S}(u_i, \alpha) = B_i - B_{i-1} = b_i$. This shows that the ordering $\mathbf{v}_\alpha$ coincides with the ordering $\mathbf{v}$ and $\mathrm{sh}(\alpha) = (b_1, b_2, \ldots, b_{n-1})$. □

Remark 5. Some of the matchings F_j^0 constructed in the first part of the proof of Equivalence lemma may be splitted perfect matchings as well, but for each of them both their left and right parts have the same color in the coloring α. For example, in case $|\mathrm{sh}(\alpha)| = 0$, $F^0_{\alpha(u_1v_1)} = C_{\alpha(u_1v_1)}(\alpha)$ is 1-splitted perfect matching with respect to the ordering $\mathbf{v}_\alpha$.

Corollary 1. *For any $n \in \mathbb{N}$, K_{2n} has an interval t-coloring if and only if it has a 1-factorization, where at least $t-2n+1$ perfect matchings are splitted.*

Proof. Construction of the desired 1-factorization from the interval t-coloring immediately follows from Remark 2 and Equivalence lemma. Remark 5 implies that the number of the splitted perfect matchings in the obtained 1-factorization can be more than $t-2n+1$.

If we have a 1-factorization of K_{2n} with at least $t-2n+1$ splitted perfect matchings we can arbitrarily choose exactly $t-2n+1$ of them, then for each of them choose the i for which it is i-splitted (the same perfect matching can be both i-splitted and i'-splitted for distinct i and i', the choice is again arbitrary) and apply Equivalence lemma. So, the corresponding coloring may not be uniquely determined. □

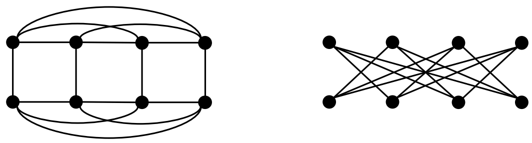

Fig. 2. Two spanning regular subgraphs of K_8.

This corollary shows that finding an interval edge-coloring of K_{2n} with many colors is equivalent to finding a 1-factorization with many splitted perfect matchings with respect to some ordering of vertices. For the ordering $\mathbf{v}$ we can define the maximum number of splitted perfect matchings over all 1-factorizations of K_{2n}. Because of the symmetry of complete graph this number does not actually depend on the chosen ordering $\mathbf{v}$, so we denote it by σ_n.

Theorem 7 (*Equivalence Theorem*)**.** *For every $n \in \mathbb{N}$, $W(K_{2n}) = 2n - 1 + \sigma_n$.*

3. Lower bounds

In order to obtain new lower bounds on $W(K_{2n})$ we split K_{2n} into two edge-disjoint spanning regular subgraphs, find convenient 1-factorizations for each of them, and then apply Equivalence theorem for the union of these 1-factorizations.

We fix the ordering of vertices of K_{2n}, $\mathbf{v} = (u_1, v_1, u_2, v_2, \dots, u_n, v_n)$, and define two spanning regular subgraphs of K_{2n}, $K_2\square K_n$ and $K_2 \times K_n$ (Fig. 2):

$$V(K_2\square K_n) = V(K_2 \times K_n) = V(K_{2n})$$
$$E(K_2\square K_n) = \{u_iu_j \mid 1 \le i < j \le n\} \cup \{u_iv_i \mid 1 \le i \le n\} \cup \{v_iv_j \mid 1 \le i < j \le n\}$$
$$E(K_2 \times K_n) = \{u_iv_j \mid 1 \le i \ne j \le n\}.$$

Note that $E(K_{2n}) = E(K_2\square K_n) \cup E(K_2 \times K_n)$. We fix an ordering of vertices $\mathbf{v} = (u_1, v_1, u_2, v_2, \dots, u_n, v_n)$ and define a special 1-factorization of $K_2\square K_n$ which we denote by $\mathfrak{P}_n$:

$\mathfrak{P}_n = \{P_0, P_1, \dots, P_{n-1}\}$, where

$$P_0 = \begin{cases} \left\{u_ju_{n+1-j}, v_jv_{n+1-j} \mid j = 1, 2, \dots, \dfrac{n}{2}\right\} & \text{if } n \text{ is even} \\ \left\{u_ju_{n+1-j}, v_jv_{n+1-j} \mid j = 1, 2, \dots, \left\lfloor \dfrac{n}{2} \right\rfloor\right\} \cup \{u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\}, & \text{if } n \text{ is odd.} \end{cases}$$

For every $i = 1, 2, \dots, n-1$, $P_i = l^i_{\mathbf{v}}(P_i) \cup r^i_{\mathbf{v}}(P_i)$, (see Fig. 3) where

$$l^i_{\mathbf{v}}(P_i) = \begin{cases} \left\{u_ju_{i+1-j}, v_jv_{i+1-j} \mid j = 1, 2, \dots, \dfrac{i}{2}\right\} & \text{if } i \text{ is even} \\ \left\{u_ju_{i+1-j}, v_jv_{i+1-j} \mid j = 1, 2, \dots, \left\lfloor \dfrac{i}{2} \right\rfloor\right\} \cup \{u_{\frac{i+1}{2}} v_{\frac{i+1}{2}}\}, & \text{if } i \text{ is odd} \end{cases}$$

$$r^i_{\mathbf{v}}(P_i) = \begin{cases} \left\{u_{i+j}u_{n+1-j}, v_{i+j}v_{n+1-j} \mid j = 1, 2, \dots, \dfrac{n-i}{2}\right\} & \text{if } n-i \text{ is even} \\ \left\{u_{i+j}u_{n+1-j}, v_{i+j}v_{n+1-j} \mid j = 1, 2, \dots, \left\lfloor \dfrac{n-i}{2} \right\rfloor\right\} \cup \{u_{\frac{n+i+1}{2}} v_{\frac{n+i+1}{2}}\}, & \text{if } n-i \text{ is odd.} \end{cases}$$

P_i is clearly an i-splitted perfect matching, for every $i = 1, 2, \dots, n-1$. Note, that $K_2 \times K_n$ is a regular bipartite graph, so König's theorem [9] implies it has a 1-factorization. If we consider the perfect matchings of any 1-factorization of $K_2 \times K_n$ as non-splitted matchings and add the perfect matchings of $\mathfrak{P}_n$ we obtain that $\sigma_n \ge n - 1$. Equivalence theorem implies that this result is equivalent to Theorem 4.

In order to improve this bound we concentrate on finding a better 1-factorization of $K_2 \times K_n$.

Lemma 3. *If $n \ge 2$, then $\sigma_n \ge \lfloor 1.5n \rfloor - 2$.*

Proof. We fix an ordering of vertices $\mathbf{v} = (u_1, v_1, u_2, v_2, \dots, u_n, v_n)$ and consider two induced subgraphs:

$$G_1 = K_2 \times K_n\left[\left\{u_1, v_1, u_2, v_2, \dots, u_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor}\right\}\right]$$
$$G_2 = K_2 \times K_n\left[\left\{u_{\lfloor \frac{n}{2} \rfloor+1}, v_{\lfloor \frac{n}{2} \rfloor+1}, u_{\lfloor \frac{n}{2} \rfloor+2}, v_{\lfloor \frac{n}{2} \rfloor+2}, \dots, u_n, v_n\right\}\right].$$

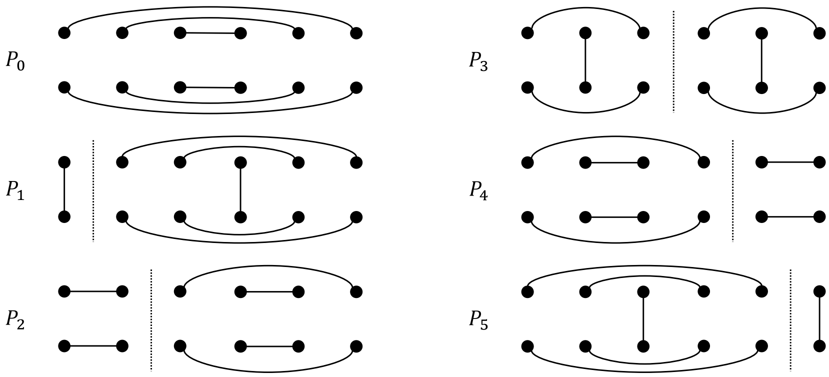

Fig. 3. 1-factorization $\mathfrak{P}_6$ of $K_2\square K_6$.

Both subgraphs are regular and bipartite, so according to the König's theorem [9] they have 1-factorizations. Let the 1-factorizations of G_1 and G_2 be $F_1^l, F_2^l, \ldots, F^l_{\lfloor\frac{n}{2}\rfloor-1}$ and $F_1^r, F_2^r, \ldots, F^r_{\lceil\frac{n}{2}\rceil-1}$, respectively. By joining the first $\lfloor\frac{n}{2}\rfloor-1$ pairs of these matchings we form $\lfloor\frac{n}{2}\rfloor$-splitted perfect matchings of $K_2 \times K_n$ with respect to the ordering $\mathbf{v}$:

$$F_i = F_i^l \cup F_i^r, \text{ for all } i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

If we remove the edges $\bigcup_{i=1}^{\lfloor\frac{n}{2}\rfloor-1} F_i$ from the graph $K_2 \times K_n$, the remaining graph is still a regular bipartite graph and has a 1-factorization, which we denote by $\mathfrak{F}_0$. Now, $\mathfrak{F}_0 \cup \bigcup_{i=1}^{\lfloor\frac{n}{2}\rfloor-1} F_i \cup \mathfrak{P}_n$ is a 1-factorization of K_{2n}. The number of splitted matchings is $\lfloor\frac{n}{2}\rfloor - 1 + n - 1$. So we have $\sigma_n \geq \lfloor 1.5n\rfloor - 2$. $\square$

By applying Equivalence theorem we obtain the following lower bound:

Theorem 8. *If $n \geq 2$, then $W(K_{2n}) \geq \lfloor 3.5n\rfloor - 3$.*

This theorem implies that $W(K_{10}) \geq 14$ which is the smallest example that disproves Conjecture 1. Next we focus on the case when n is a composite number.

Lemma 4. *For any $m, n \in \mathbb{N}$, $\sigma_{mn} \geq \sigma_m + \sigma_n + 2(m-1)(n-1)$.*

Proof. Let the vertex sets of K_{2mn}, K_{2n} and K_{2m} be as follows:

$$\begin{aligned}
V(K_{2mn}) &= \left\{u_i^j, v_i^j \mid i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m\right\}\\
V(K_{2n}) &= \{\overline{u}_i, \overline{v}_i \mid i = 1, 2, \ldots, n\}\\
V(K_{2m}) &= \left\{\widetilde{u}^i, \widetilde{v}^i \mid i = 1, 2, \ldots, m\right\}.
\end{aligned}$$

We fix the following orderings of vertices of K_{2mn}, K_{2n} and K_{2m}, respectively:

$$\begin{aligned}
\mathbf{v} &= \left(u_1^1, v_1^1, u_2^1, v_2^1, \ldots, u_n^1, v_n^1, u_1^2, v_1^2, u_2^2, v_2^2, \ldots, u_n^2, v_n^2, \ldots, u_1^m, v_1^m, u_2^m, v_2^m, \ldots, u_n^m, v_n^m\right)\\
\overline{\mathbf{v}} &= (\overline{u}_1, \overline{v}_1, \overline{u}_2, \overline{v}_2, \ldots, \overline{u}_n, \overline{v}_n)\\
\widetilde{\mathbf{v}} &= \left(\widetilde{u}^1, \widetilde{v}^1, \widetilde{u}^2, \widetilde{v}^2, \ldots, \widetilde{u}^m, \widetilde{v}^m\right).
\end{aligned}$$

Let $\overline{\mathfrak{F}} = \{N_1, N_2, \ldots, N_{\sigma_n}, N_1^0, N_2^0, \ldots, N^0_{2n-1-\sigma_n}\}$ be a 1-factorization of K_{2n}, where $N_i, i = 1, 2, \ldots, \sigma_n$ are splitted perfect matchings. Let $\widetilde{\mathfrak{F}} = \{M_1, M_2, \ldots, M_{\sigma_m}, M_1^0, M_2^0, \ldots, M^0_{2m-1-\sigma_m}\}$ be a 1-factorization of K_{2m}, where M_i, $i = 1, 2, \ldots, \sigma_m$ are splitted perfect matchings.

We also need the graph $K_2\square K_{2m}$ with the vertex set $\{w_i, z_i \mid i = 1, 2, \ldots, 2m\}$, an ordering of its vertices $\mathbf{w} = (w_1, z_1, w_2, z_2, \ldots, w_{2m}, z_{2m})$, and its 1-factorization $\mathfrak{P}_{2m} = \{P_0, P_1, \ldots, P_{2m-1}\}$ as defined at the beginning of this section. We call the subgraph $K_2\square K_{2m}[\{w_{2k-1}, w_{2k}, z_{2k-1}, z_{2k}\}]$ kth cell of $K_2\square K_{2m}$, $1 \leq k \leq m$.

During the proof we always assume that $x, y \in \{u, v\}$, $1 \leq s, t \leq n$ and $1 \leq p, q \leq m$.

Let $\overline{\varphi}$ be a mapping which projects the edges of K_{2mn} to the edges of K_{2n}. For every edge $x_s^p y_t^q \in E(K_{2mn})$, where $x_s \neq y_t$, we define $\overline{\varphi}(x_s^p y_t^q) = \overline{x}_s\overline{y}_t$. Next we define a mapping $\widetilde{\varphi}$ which projects the remaining edges of K_{2mn} to the edges of K_{2m}. For every edge $x_s^p x_s^q \in E(K_{2mn})$ we define $\widetilde{\varphi}(x_s^p x_s^q) = \widetilde{x}^p\widetilde{x}^q$. Note that the preimages $\overline{\varphi}^{-1}(\overline{e})$ for all $\overline{e} \in E(K_{2n})$ and $\widetilde{\varphi}^{-1}(\widetilde{x}^p\widetilde{x}^q)$ for

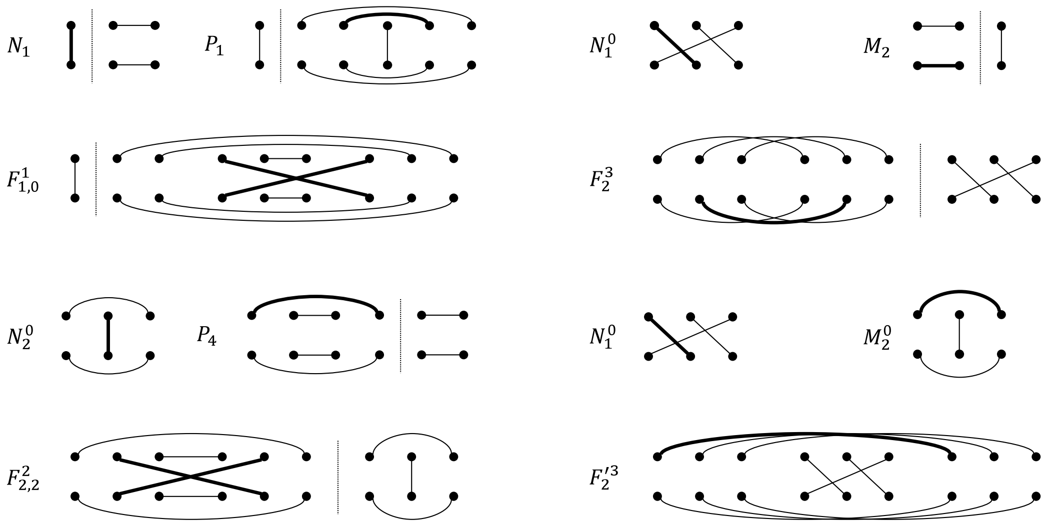

Fig. 4. Several perfect matchings of K_{18} constructed based on 1-factorizations $\overline{\mathfrak{F}} = \{N_1, N_2, N_1^0, N_2^0, N_3^0\}$ of K_6, $\widetilde{\mathfrak{F}} = \{M_1, M_2, M_1^0, M_2^0, M_3^0\}$ of K_6 and $\mathfrak{P}_6 = \{P_0, P_1, P_2, P_3, P_4, P_5\}$ of $K_2 \square K_6$ using Lemma 4.

all $\widetilde{x}^p\widetilde{x}^q \in E(K_{2m})$ are pairwise disjoint and their union covers the set $E(K_{2mn})$. We split the edge set $E(K_{2mn})$ into three parts the following way:

$$E(K_{2mn}) = E^1 \cup E^2 \cup E^3, \text{ where}$$

$$E^1 = \bigcup_{i=1}^{\sigma_n} \bigcup_{\overline{e} \in N_i} \overline{\varphi}^{-1}(\overline{e})$$

$$E^2 = \bigcup_{i=2}^{2n-1-\sigma_n} \bigcup_{\overline{e} \in N_i^0} \overline{\varphi}^{-1}(\overline{e})$$

$$E^3 = \bigcup_{\overline{e} \in N_1^0} \overline{\varphi}^{-1}(\overline{e}) \cup \bigcup_{\widetilde{x}^p\widetilde{x}^q \in E(K_{2m})} \widetilde{\varphi}^{-1}(\widetilde{x}^p\widetilde{x}^q).$$

The 1-factorization of K_{2mn} we are going to construct is denoted by $\mathfrak{F}$ and also consists of three parts.

$$\mathfrak{F} = \mathfrak{F}^1 \cup \mathfrak{F}^2 \cup \mathfrak{F}^3.$$

The set of perfect matchings $\mathfrak{F}^k$ covers the set E^k, $k = 1, 2, 3$. Fig. 4 displays example perfect matchings for each of the parts in case $m = n = 3$.

The set E^1 contains the preimages of splitted perfect matchings of K_{2n}. To cover it, for every splitted perfect matching $N_i \in \overline{\mathfrak{F}}$, $i = 1, 2, \dots, \sigma_n$, and for every perfect matching with an odd index $P_{2j+1} \in \mathfrak{P}_{2m}$, $j = 0, 1, \dots, m-1$, we construct one perfect matching of $\mathfrak{F}^1$.

$$F_{i,j}^1 = F_{i,j,1}^1 \cup F_{i,j,2}^1 \cup F_{i,j,3}^1 \cup F_{i,j,4}^1, \text{ where}$$

$$F_{i,j,1}^1 = \bigcup_{\substack{w_{2k-1}z_{2k-1} \in P_{2j+1} \\ 1 \le k \le m}} \left\{ x_s^k y_t^k \mid \overline{x}_s\overline{y}_t \in l(N_i) \right\}$$

$$F_{i,j,2}^1 = \bigcup_{\substack{w_{2k}z_{2k} \in P_{2j+1} \\ 1 \le k \le m}} \left\{ x_s^k y_t^k \mid \overline{x}_s\overline{y}_t \in r(N_i) \right\}$$

$$F_{i,j,3}^1 = \bigcup_{\substack{w_{2k-1}w_{2l-1} \in P_{2j+1} \\ 1 \le k < l \le m}} \left\{ x_s^k y_t^l, y_t^k x_s^l \mid \overline{x}_s\overline{y}_t \in l(N_i) \right\}$$

$$F_{i,j,4}^1 = \bigcup_{\substack{w_{2k}w_{2l} \in P_{2j+1} \\ 1 \le k < l \le m}} \left\{ x_s^k y_t^l, y_t^k x_s^l \mid \overline{x}_s\overline{y}_t \in r(N_i) \right\}$$

$$\mathfrak{F}^1 = \left\{ F_{i,j}^1 \mid i = 1, 2, \dots, \sigma_n,\ j = 0, 1, \dots, m-1 \right\}.$$

For $F^1_{i,j,1}$ and $F^1_{i,j,2}$, we look for vertical edges in P_{2j+1}. If for some k, the vertical edge of the left (right) part of the kth cell belongs to P_{2j+1}, we add the preimages of all edges of $l(N_i)$ $(r(N_i))$ in the kth copy of K_{2n} in K_{2mn} to $F^1_{i,j,1}$ $(F^1_{i,j,2})$. Every matching P_{2j+1} contains exactly two vertical edges ($w_{j+1}z_{j+1}$ and $w_{j+m+1}z_{j+m+1}$). If m is odd, then one of these two belongs to the left part of its cell, and the other one belongs to the right part of its cell. If m is even, then if j is odd (even), both vertical edges belong to the right (left) parts of the cells. So, the number of edges in $F^1_{i,j,1}$ and $F^1_{i,j,2}$ can be calculated the following way:

$$|F^1_{i,j,1}| = |l(N_i)| \left((m \bmod 2) \cdot 1 + (1 - m \bmod 2) \cdot 2(1 - j \bmod 2)\right)$$
$$|F^1_{i,j,2}| = |r(N_i)| \left((m \bmod 2) \cdot 1 + (1 - m \bmod 2) \cdot 2(j \bmod 2)\right).$$

For $F^1_{i,j,3}$ $(F^1_{i,j,4})$ we are looking for edges joining left side (right side) vertices of two different cells in P_{2j+1}. If m is odd, then there are $\frac{m-1}{2}$ such edges. If m is even, then there are $\frac{m}{2} - (1 - j \bmod 2)$ (in case of $F^1_{i,j,4}$: $\frac{m}{2} - (j \bmod 2)$) such edges. For every such edge which joins the kth and lth cells ($k < l$) we add the preimages of all edges in $l(N_i)$ $(r(N_i))$ which join the vertices in kth and lth copies of K_{2n} in K_{2mn} to $F^1_{i,j,3}$ $(F^1_{i,j,4})$. Note that for every chosen edge from P_{2j+1}, every edge in $l(N_i)$ $(r(N_i))$ has exactly 2 preimages in $F^1_{i,j,3}$ $(F^1_{i,j,4})$. So we have:

$$|F^1_{i,j,3}| = 2|l(N_i)| \left((m \bmod 2) \cdot \frac{m-1}{2} + (1 - m \bmod 2) \cdot \left(\frac{m}{2} - (1 - j \bmod 2)\right)\right)$$
$$|F^1_{i,j,4}| = 2|r(N_i)| \left((m \bmod 2) \cdot \frac{m-1}{2} + (1 - m \bmod 2) \cdot \left(\frac{m}{2} - (j \bmod 2)\right)\right).$$

The construction of $F^1_{i,j}$ implies that it is a matching in K_{2mn}. To prove that it is also a perfect matching, we need to show that it has exactly mn edges.

$$\begin{aligned}
|F^1_{i,j}| &= |F^1_{i,j,1}| + |F^1_{i,j,2}| + |F^1_{i,j,3}| + |F^1_{i,j,4}| \\
&= |l(N_i)| \left((m \bmod 2)(1 + m - 1) + (1 - m \bmod 2)\left(2(1 - j \bmod 2) + m - 2(1 - j \bmod 2)\right)\right) \\
&\quad + |r(N_i)| \left((m \bmod 2)(1 + m - 1) + (1 - m \bmod 2)\left(2(j \bmod 2) + m - 2(j \bmod 2)\right)\right) \\
&= (|l(N_i)| + |r(N_i)|) \left((m \bmod 2) \cdot m + (1 - m \bmod 2) \cdot m\right) = nm.
\end{aligned}$$

The matchings $F^1_{i,j}$ and $F^1_{i',j'}$ are disjoint if $i \neq i'$ or $j \neq j'$, as their edges correspond to either different edges in K_{2n} or to different edges in $K_2 \square K_{2m}$. Also note that, if N_i is an r-splitted matching for $\overline{\mathbf{v}}$, then $F^1_{i,j}$ is $(jn + r)$-splitted matching for $\mathbf{v}$, for every $i = 1, 2, \ldots, \sigma_n$ and $j = 0, 1, \ldots, m - 1$.

The set E^2 contains the preimages of all but one non-splitted perfect matchings. To cover it, for every non-splitted perfect matching $N^0_i \in \overline{\mathfrak{F}}$ except N^0_1 (the choice of this exception is arbitrary) and for every perfect matching with an even index $P_{2j} \in \mathfrak{P}_{2m}$ we construct one perfect matching of $\mathfrak{F}^2$.

$$F^2_{i,j} = F^2_{i,j,1} \cup F^2_{i,j,2}, \text{ where}$$
$$F^2_{i,j,1} = \bigcup_{\substack{w_{2k-1}w_{2k} \in P_{2j} \\ 1 \le k \le m}} \left\{x^k_s y^k_t \mid \bar{x}_s \bar{y}_t \in N^0_i\right\}$$
$$F^2_{i,j,2} = \bigcup_{\substack{w_{2k-1}w_{2l} \in P_{2j} \\ 1 \le k < l \le m}} \left\{x^k_s y^l_t, y^k_t x^l_s \mid \bar{x}_s \bar{y}_t \in N^0_i\right\}$$
$$\mathfrak{F}^2 = \{F^2_{i,j} \mid i = 2, 3, \ldots, 2n - 1 - \sigma_n,\ j = 0, 1, \ldots, m - 1\}.$$

The matchings P_{2j} have only horizontal edges. We look for those edges which join a vertex from the left part of a cell to a vertex from the right part of a (possibly different) cell. If both endpoints of an edge belong to the same kth cell, we add the preimages of all edges of N^0_i which belong to the kth copy of K_{2n} in K_{2mn} to the set $F^2_{i,j,1}$. The number of such edges in P_{2j} is 1 if m is odd and $2(j \bmod 2)$ if m is even. So we have:

$$|F^2_{i,j,1}| = n \left((m \bmod 2) \cdot 1 + (1 - m \bmod 2) \cdot 2(j \bmod 2)\right).$$

If the edge of P_{2j} joins vertices of kth and lth cells ($k < l$) then we add both preimages of all edges of N^0_i which join the vertices of kth and lth copies of K_{2n} in K_{2mn} to $F^2_{i,j,2}$. The number of such edges in P_{2j} is $\frac{m-1}{2}$ if m is odd, and $\frac{m}{2} - (j \bmod 2)$ if m is even. So,

$$|F^2_{i,j,2}| = 2n \left((m \bmod 2) \cdot \frac{m-1}{2} + (1 - m \bmod 2) \cdot \left(\frac{m}{2} - (j \bmod 2)\right)\right)$$
$$|F^2_{i,j}| = |F^2_{i,j,1}| + |F^2_{i,j,2}|$$

$$= n\left((m \bmod 2)(1+m-1) + (1-m \bmod 2)(2(j \bmod 2) + m - 2(j \bmod 2))\right)$$
$$= n\left((m \bmod 2)\cdot m + (1-m \bmod 2)\cdot m\right) = nm.$$

Similar to the matchings in $\mathfrak{F}^1$, the matchings $F^2_{i,j}$ and $F^2_{i',j'}$ are disjoint if $i \neq i'$ or $j \neq j'$. Note that for every $i = 2, 3, \ldots, 2n-1-\sigma_n$, $F^2_{i,j}$ is jn-splitted perfect matching for $\mathbf{v}$ for every $j = 1, 2, \ldots, m-1$, and is a non-splitted perfect matching if $j = 0$.

The set E^3 contains the preimages of the edges of the non-splitted perfect matching N^0_1 of K_{2n} and the preimages of all edges of $K_{2m}[\{\widetilde{u}^1, \widetilde{u}^2, \ldots, \widetilde{u}^m\}] \cup K_{2m}[\{\widetilde{v}^1, \widetilde{v}^2, \ldots, \widetilde{v}^m\}]$. The preimages of the edges of K_{2m} form $2n$ disjoint complete graphs on m vertices, namely $K_{2mn}\left[\{x^1_s, x^2_s, \ldots, x^m_s\}\right]$, for every $\bar{x}_s \in V(K_{2n})$. For every edge $\bar{x}_s\bar{y}_t \in N^0_1$, its preimages together with the two copies of K_m corresponding to the vertices $\bar{x}_s$ and $\bar{y}_t$ form the subgraph $K_{2mn}\left[\{x^1_s, y^1_t, x^2_s, y^2_t, \ldots, x^m_s, y^m_t\}\right]$, which is isomorphic to K_{2m}. So, the set E^3 consists of n disjoint copies of K_{2m}. For every perfect matching $M \in \widetilde{\mathfrak{F}}$ we construct one perfect matching in K_{2mn} by joining its n disjoint copies in E^3:

$$F^3_i = \bigcup_{\bar{x}_s\bar{y}_t \in N^0_1} \left\{\{x^p_s x^q_s \mid \widetilde{u}^p\widetilde{u}^q \in M_i\} \cup \{x^p_s y^q_t \mid \widetilde{u}^p\widetilde{v}^q \in M_i\} \cup \{y^p_t y^q_t \mid \widetilde{v}^p\widetilde{v}^q \in M_i\}\right\}$$

$$F'^3_i = \bigcup_{\bar{x}_s\bar{y}_t \in N^0_1} \left\{\{x^p_s x^q_s \mid \widetilde{u}^p\widetilde{u}^q \in M^0_i\} \cup \{x^p_s y^q_t \mid \widetilde{u}^p\widetilde{v}^q \in M^0_i\} \cup \{y^p_t y^q_t \mid \widetilde{v}^p\widetilde{v}^q \in M^0_i\}\right\}$$

$$\mathfrak{F}^3 = \left\{F^3_i \mid i = 1, 2, \ldots, \sigma_m\right\} \cup \left\{F'^3_i \mid i = 1, 2, \ldots, 2m-1-\sigma_m\right\}.$$

The sets F^3_i and F'^3_i are pairwise disjoint matchings having mn edges each. Note that if M_i is r-splitted perfect matching for $\widetilde{\mathbf{v}}$, then F^3_i is rn-splitted perfect matching for $\mathbf{v}$, $i = 1, 2, \ldots, \sigma_m$. Moreover, the perfect matchings F'^3_i are not splitted, $i = 1, 2, \ldots, 2m-1-\sigma_m$.

The number of the constructed perfect matchings in $\mathfrak{F}$ is $m\sigma_n + m(2n-2-\sigma_n) + 2m - 1 = 2mn - 1$. Out of these the number of splitted perfect matchings is $m\sigma_n + (m-1)(2n-2-\sigma_n) + \sigma_m = \sigma_m + \sigma_n + 2(m-1)(n-1)$. This completes the proof. $\square$

By applying Equivalence theorem we obtain the following lower bound, which is a generalization of Theorem 5:

Theorem 9. *For any $m, n \in \mathbb{N}$, $W(K_{2mn}) \geq W(K_{2m}) + W(K_{2n}) + 4(m-1)(n-1) - 1$.*

We know that $W(K_6) = 7$ and $W(K_{10}) \geq 14$. The above theorem implies that $W(K_{30}) \geq 52$. This result disproves Conjecture 2 which predicted that $W(K_{30}) = 51$. But this is not the smallest case that contradicts the conjecture as we will see in Section 5.

Corollary 2. *If $n = \prod_{i=1}^{\infty} p_i^{\alpha_i}$, where p_i is the ith prime number, $\alpha_i \in \mathbb{Z}_+$, then*

$$W(K_{2n}) \geq 4n - 3 - \sum_{i=1}^{\infty} \alpha_i \left(4p_i - 3 - W(K_{2p_i})\right).$$

Proof. Let d_m denote the difference $W(K_{2m}) - (4m-3)$. Theorem 9 states that $d_{mk} \geq d_m + d_k$. By induction we get $d_n \geq \sum_{i=1}^{\infty} \alpha_i d_{p_i}$. We complete the proof by replacing d_{p_i} by its value. $\square$

4. Upper bounds

Let α be an arbitrary interval edge-coloring of K_{2n}, $n \in \mathbb{N}$, and $\mathbf{v}_\alpha = (u_1, v_1, u_2, v_2, \ldots, u_n, v_n)$ be its corresponding ordering of vertices. Let the shift vector of α be $\mathrm{sh}(\alpha) = (b_1, b_2, \ldots, b_{n-1})$. Equivalence lemma implies that there exists a 1-factorization of K_{2n} $\mathfrak{F} = \left\{F^0_j \mid j = 1, 2, \ldots, 2n-1-\sum_{i=1}^{n-1} b_i\right\} \cup \bigcup_{i=1}^{n-1}\left\{F^i_j \mid j = 1, 2, \ldots, b_i\right\}$, such that F^i_j is i-splitted with respect to the ordering $\mathbf{v}_\alpha$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, b_i$. Wherever we have an interval coloring α of a complete graph in the proofs of this section we will always assume that the corresponding ordering of vertices $\mathbf{v}_\alpha$ and 1-factorization $\mathfrak{F}$ is given.

To improve the upper bounds on $W(K_{2n})$ we need several lemmas.

Lemma 5. *If for some interval edge-coloring α of K_{2n}, $\mathrm{sh}(\alpha) = (b_1, b_2, \ldots, b_{n-1})$, then there exists interval edge-coloring β of K_{2n} such that $\mathrm{sh}(\beta) = (b_{n-1}, b_{n-2}, \ldots, b_1)$.*

Proof. Note that if some $F \in \mathfrak{F}$ is i-splitted with respect to $\mathbf{v}_\alpha$, then it is $(n-i)$-splitted with respect to the ordering $\mathbf{v}'_\alpha = (u_n, v_n, u_{n-1}, v_{n-1}, \ldots, u_1, v_1)$. We use Equivalence lemma to construct a coloring β from $\mathfrak{F}$ with respect to the ordering $\mathbf{v}'_\alpha$. Its shift vector is $(b_{n-1}, b_{n-2}, \ldots, b_1)$. $\square$

Lemma 6. *If for some interval edge-coloring α of K_{2n}, $\mathrm{sh}(\alpha) = (b_1, b_2, \dots, b_{n-1})$, where $b_i > 0$ for some $i \in [1, n-1]$, then there exists interval edge-coloring β of K_{2n} such that $\mathrm{sh}(\beta) = (b_1, b_2, \dots, b_{i-1}, b_i - 1, b_{i+1}, \dots, b_{n-1})$.*

Proof. The condition $b_i > 0$ implies that there exists a perfect matching $F^i_{b_i} \in \mathfrak{F}$ which is i-splitted with respect to the ordering $\mathbf{v}_\alpha$. We construct the coloring β by applying Equivalence lemma to the 1-factorization $\mathfrak{F}$ by regarding the perfect matching $F^i_{b_i}$ as a non-splitted one (we can rename it to $F^0_{2n-|\mathrm{sh}(\alpha)|}$). $\square$

Lemma 7. *If $\mathrm{sh}(\alpha) = (b_1, b_2, \dots, b_{n-1})$ for some interval edge-coloring α of K_{2n}, then*

$$\sum_{i=1}^{k} b_i \leq 2k - 1, \quad \text{for every } k = 1, 2, \dots, n-1.$$

Proof. According to the proof of Equivalence lemma, the left parts of the perfect matchings F^i_j cover the vertex u_1 (and v_1), $i = 1, 2, \dots, k, j = 1, 2, \dots, b_i$. Moreover,

$$\bigcup_{i=1}^{k} \bigcup_{j=1}^{b_i} l^i_{\mathbf{v}_\alpha}\left(F^i_j\right) \subset E\left(H^{[1,k]}_{\mathbf{v}_\alpha}\right).$$

To complete the proof we note that the number of the perfect matchings F^i_j is $\sum_{i=1}^{k} b_i$ and the degree of the vertex u_1 (or v_1) in $H^{[1,k]}_{\mathbf{v}_\alpha}$ is $2k - 1$. $\square$

We will call the vector $(b_1, b_2, \dots, b_k)$ *saturated* if $\sum_{i=1}^{k} b_i = 2k - 1$.

Corollary 3. *If α is an interval edge-coloring of K_{2n}, $n \geq 3$, then $|\mathrm{sh}(\alpha)| \leq 2n - 4$.*

Proof. Let $\mathrm{sh}(\alpha) = (b_1, b_2, \dots, b_{n-1})$. Lemma 7 implies that $\sum_{i=1}^{n-2} b_i \leq 2n - 5$. The same lemma in conjunction with Lemma 5 implies that $b_{n-1} \leq 1$. By summing these two inequalities we complete the proof. $\square$

Lemma 8. *If $\mathrm{sh}(\alpha) = (b_1, b_2, \dots, b_{n-1})$ for some interval edge-coloring α of K_{2n} and $(b_1, b_2, \dots, b_k)$ is saturated for some $k \in [2, n-2]$, then $b_{k+1} \leq 1$.*

Proof. Lemma 7 implies that $b_{k+1} \leq 2$. To complete the proof we need to show that $b_{k+1} \neq 2$. Suppose the contrary, $b_{k+1} = 2$. $(b_1, b_2, \dots, b_k)$ is saturated, so the proof of Lemma 7 implies that the edges u_1x_i and v_1x_i, $x \in \{u, v\}$, $i = 2, 3, \dots, k$, belong to the perfect matchings F^i_j, $i = 1, 2, \dots, k, j = 1, 2, \dots, b_i$. Similarly, the edges u_1u_{k+1}, u_1v_{k+1}, v_1u_{k+1} and v_1v_{k+1} must be covered by F^{k+1}_1 and F^{k+1}_2.

Now we look at the vertex u_2. It is covered by the left parts of the perfect matchings F^i_j, $i = 2, 3, \dots, k, j = 1, 2, \dots, b_i$. In total these matchings cover all but $2k - 1 - \sum_{i=2}^{k} b_i = b_1$ edges incident to u_2 in the subgraph $H^{[1,k]}_{\mathbf{v}_\alpha}$. Lemma 7 implies that $b_1 \leq 1$, so at most one edge is left uncovered. The vertex u_2 must be covered by the left parts of F^{k+1}_1 and F^{k+1}_2 as well. The edges u_2u_{k+1} and u_2v_{k+1} cannot be used as the vertices u_{k+1} and v_{k+1} are already covered by F^{k+1}_1 and F^{k+1}_2. Therefore, at most one edge remains for these two matchings, which is a contradiction. $\square$

Corollary 4. *If α is an interval edge-coloring of K_{2n}, $n \geq 5$, then $|\mathrm{sh}(\alpha)| \leq 2n - 5$.*

Proof. Let $\mathrm{sh}(\alpha) = (b_1, b_2, \dots, b_{n-1})$. Lemma 7 implies that $\sum_{i=1}^{n-3} b_i \leq 2n - 7$. We consider two cases.

Case 1: $\sum_{i=1}^{n-3} b_i = 2n - 7$. Lemma 8 implies that $b_{n-2} \leq 1$. Lemmas 5 and 7 imply that $b_{n-1} \leq 1$. The sum of these inequalities proves the required bound.

Case 2: $\sum_{i=1}^{n-3} b_i \leq 2n - 8$. Lemmas 5 and 7 imply that $b_{n-2} + b_{n-1} \leq 3$. The sum of these inequalities completes the proof. $\square$

Lemma 9. *If $\mathrm{sh}(\alpha) = (b_1, b_2, \dots, b_{n-1})$ for some interval edge-coloring α of K_{2n} and $(b_1, b_2, \dots, b_k)$ is saturated for some $k \in [3, n-1]$, then $b_k \geq 3$.*

Proof. Suppose the contrary, $b_k \leq 2$. If $b_k = 2$, then the vector $(b_1, b_2, \dots, b_{k-1})$ is also saturated, and we obtain contradiction with Lemma 8. If $b_k \leq 1$, then we have $\sum_{i=1}^{k-1} b_i \geq 2k - 2$ which contradicts Lemma 7. $\square$

Lemma 10. *If $\mathrm{sh}(\alpha) = (b_1, b_2, \dots, b_{n-1})$ for some interval edge-coloring α of K_{2n} and $k \in [2, n-2]$, then*

$$k(2k - 1) \geq \sum_{i=1}^{k} ib_i + \sum_{i=k+1}^{\min\{2k-1, n-1\}} (2k - i)b_i.$$

Proof. We consider the subgraph $H^{[1,k]}_{\mathbf{v}_\alpha}$. The number of edges in the subgraph is $k(2k-1)$. The left part of each of the perfect matchings F^i_j, $i = 1, 2, \dots, k, j = 1, 2, \dots, b_i$, consists of i edges, and all of them belong to the subgraph $H^{[1,k]}_{\mathbf{v}_\alpha}$. The number of such edges is $\sum_{i=1}^{k} ib_i$.

Table 1
The values of $m(k, r)$. The first row of each of the cells displays the value of $m(k, r)$. The second row contains some vector $(b_1, b_2, \ldots, b_k) \in T_k$ for which $\sum_{i=1}^{k} ib_i = m(k, r)$.

r	k			
	1	2	3	4
0	0 (0)	0 (0, 0)	0 (0, 0, 0)	0 (0, 0, 0, 0)
1	1 (1)	1 (1, 0)	1 (1, 0, 0)	1 (1, 0, 0, 0)
2		3 (1, 1)	3 (1, 1, 0)	3 (1, 1, 0, 0)
3		5 (1, 2)	5 (1, 2, 0)	5 (1, 2, 0, 0)
4			8 (1, 2, 1)	8 (1, 2, 1, 0)
5			12 (1, 1, 3)	12 (1, 2, 1, 1)
6				16 (1, 2, 1, 2)
7				20 (1, 2, 1, 3)

Now we fix an $i \in [k+1, r]$, where r denotes $\min\{2k-1, n-1\}$. The left part of each of the perfect matchings F_j^i, $j = 1, 2, \ldots, b_i$, consists of i edges. At most $2i - 2k$ of them can join some vertex from $H_{\mathbf{v}_\alpha}^{[1,k]}$ with some vertex from $H_{\mathbf{v}_\alpha}^{[k+1,i]}$. So at least $2k - i$ edges belong to the subgraph $H_{\mathbf{v}_\alpha}^{[1,k]}$. The number of such edges is at least $\sum_{i=k+1}^{r}(2k-i)b_i$. $\square$

Lemma 10 implies that if for some fixed k_0 there are many i-splitted perfect matchings where $i \le k_0$, then there cannot be too many i'-splitted perfect matchings where $k_0 < i' < \min\{2k_0 - 1, n-1\}$. In order to use this lemma we need to bound the sum $\sum_{i=1}^{k} ib_i$ from below.

For the numbers $k \in \mathbb{N}$ and $r \in \mathbb{Z}_+$ we define the following:

$$T_k = \{(b_1, b_2, \ldots, b_k) \mid \exists\alpha \text{ interval coloring of } K_{2n},\ n > k,\ \mathrm{sh}(\alpha) = (b_1, b_2, \ldots, b_{n-1})\}$$

$$m(k, r) = \min_{(b_1, b_2, \ldots, b_k) \in T_k} \left\{ \sum_{i=1}^{k} ib_i \mid \sum_{i=1}^{k} b_i = r \right\}.$$

Note that $m(k, r)$ is not defined for all pairs (k, r). For example, Lemma 7 implies that there are no interval colorings of K_{2n} for which $\sum_{i=1}^{k} b_i = r$ if $r \ge 2k$. It is obvious that $m(1, 1) = 1$ and $m(k, 0) = 0$, $k \in \mathbb{N}$.

Remark 6. In order to calculate $m(k, r)$, $k > 1$, $r > 1$, it is sufficient to take the minimum over those $(b_1, b_2, \ldots, b_k) \in T_k$ for which $\sum_{i=1}^{k-1} ib_i = m(k-1, r-b_k)$.

Table 1 lists the values of $m(k, r)$ for $k \le 4$ and $r \le 7$. For example, $m(3, 5)$ is calculated as follows. According to the above remark the possible candidate vectors from T_3 are $(1, 2, 2)$, $(1, 1, 3)$, $(1, 0, 4)$ and $(0, 0, 5)$. Lemma 8 implies that $(1, 2, 2) \notin T_3$. The coloring of K_{12} in Fig. 5 proves that $(1, 1, 3) \in T_3$. On the other hand, sum $b_1 + 2b_2 + 3b_3$ is larger for the other two candidate vectors, so $m(3, 5) = 12$. Similarly we show that $m(4, 7) = 20$ and the minimum is achieved on the vector $(1, 2, 1, 3)$, which clearly belongs to T_4 as illustrated in the coloring of K_{22} in Fig. 6. By applying Lemma 6 to these two colorings we prove that all the other vectors listed in the Table 1 belong to the corresponding T_k's.

Lemma 11. *If α is an interval edge-coloring of K_{2n}, $n \ge 9$, then $|\mathrm{sh}(\alpha)| \le 2n - 6$.*

Proof. Suppose the contrary, $|\mathrm{sh}(\alpha)| \ge 2n - 5$. Lemmas 5 and 7 imply that $\sum_{i=5}^{n-1} b_i \le 2n - 11$. We consider three cases.

Case 1: $\sum_{i=5}^{n-1} b_i = 2n - 11$. Lemmas 5, 8 and 9 imply that $b_5 \ge 3$ and $b_4 \le 1$. We apply Lemma 7 for $k = 3$ to show that $b_1 + b_2 + b_3 = 5$ and $b_4 = 1$. Then we apply Lemma 10 for $k = 3$. The left part of the inequality is 15. On the right side we have $\sum_{i=1}^{3} ib_i \ge m(3, 5) = 12$ and $\sum_{i=4}^{5}(6-i)b_i \ge 5$. These inequalities contradict Lemma 10.

Case 2: $\sum_{i=5}^{n-1} b_i = 2n - 12$. Lemma 7 implies that (b_1, b_2, b_3, b_4) is saturated. Lemma 8 implies that $b_5 \le 1$. Therefore, $(b_{n-1}, b_{n-2}, \ldots, b_6)$ is saturated and $b_5 = 1$. Lemma 9 implies that $b_6 \ge 3$. Now we apply Lemma 10 for $k = 4$. The left part of the inequality is 28. On the right side, $\sum_{i=1}^{4} ib_i \ge m(4, 7) = 20$ and $\sum_{i=5}^{7}(8-i)b_i \ge 9$. These inequalities contradict Lemma 10.

Case 3: $\sum_{i=5}^{n-1} b_i \le 2n - 13$. Lemma 7 implies that $\sum_{i=1}^{4} b_i \le 7$. By summing these two inequalities we obtain a contradiction. $\square$

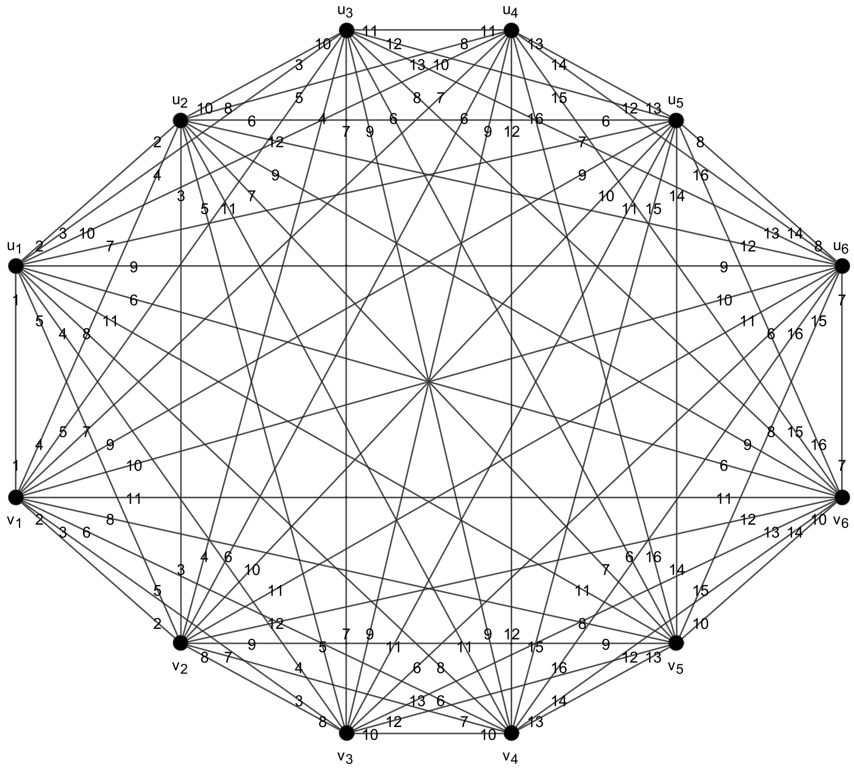

Fig. 5. Interval 16-coloring of K_{12} with a shift vector $(1, 1, 3, 0, 0)$.

Corollaries 3, 4, Lemma 11 and Remark 2 imply the following upper bound on $W(K_{2n})$.

Theorem 10. *If $n \geq 3$, then*

$$W(K_{2n}) \leq \begin{cases} 4n-5, & \text{if } n \geq 3, \\ 4n-6, & \text{if } n \geq 5, \\ 4n-7, & \text{if } n \geq 9. \end{cases}$$

5. More exact values and an improved lower bound

The lower bound on $W(K_{2n})$ from Corollary 2 depends on the values $W(K_{2p})$ where p is a prime number. For $p = 2$ and $p = 3$ the exact values of $W(K_{2p})$ were known before [11]. For $p = 5$ the lower bound from Theorem 8 coincides with the upper bound from Theorem 10. The case $p = 7$ is resolved by the lemma below. Finally, for the case $p = 11$, the upper bound from Theorem 10 is achieved by the interval 37-coloring of K_{22} shown in Fig. 6. This coloring also rejects Conjecture 2, which predicts that $W(K_{22}) = 36$.

Lemma 12. $W(K_{14}) = 21$.

Proof. Theorem 10 implies that $W(K_{14}) \leq 22$. It is sufficient to show that K_{14} does not have an interval coloring with 22 colors. Suppose the contrary, there exists α interval 22-coloring of K_{14}.

Consider its shift vector $\mathrm{sh}(\alpha) = (b_1, b_2, b_3, b_4, b_5, b_6)$. From Remark 2 we have that $\sum_{i=1}^{6} b_i = 9$. Lemma 7 implies that the sums of both first and last triples cannot exceed 5. Without loss of generality we can assume that $b_1 + b_2 + b_3 = 5$ and $b_4 + b_5 + b_6 = 4$. Lemma 8 implies that $b_4 \leq 1$. Lemmas 5 and 7 imply that $b_5 + b_6 = 3$ and $b_4 = 1$. So $b_5 \geq 2$.

Now we check the inequality from Lemma 10 for $k = 3$. The left part equals 15. On the right part we have $\sum_{i=1}^{3} ib_i \geq m(3, 5) = 12$, $\sum_{i=4}^{5}(6 - i)b_i \geq 4$. By summing these two inequalities we get a contradiction. $\square$

The best lower bound we could obtain is the following.

Theorem 11. *If $n = \prod_{i=1}^{\infty} p_i^{\alpha_i}$, where p_i is the ith prime number and $\alpha_i \in \mathbb{Z}_+$, then*

$$W(K_{2n}) \geq 4n - 3 - \alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 4\alpha_5 - \frac{1}{2}\sum_{i=6}^{\infty} \alpha_i(p_i + 1).$$

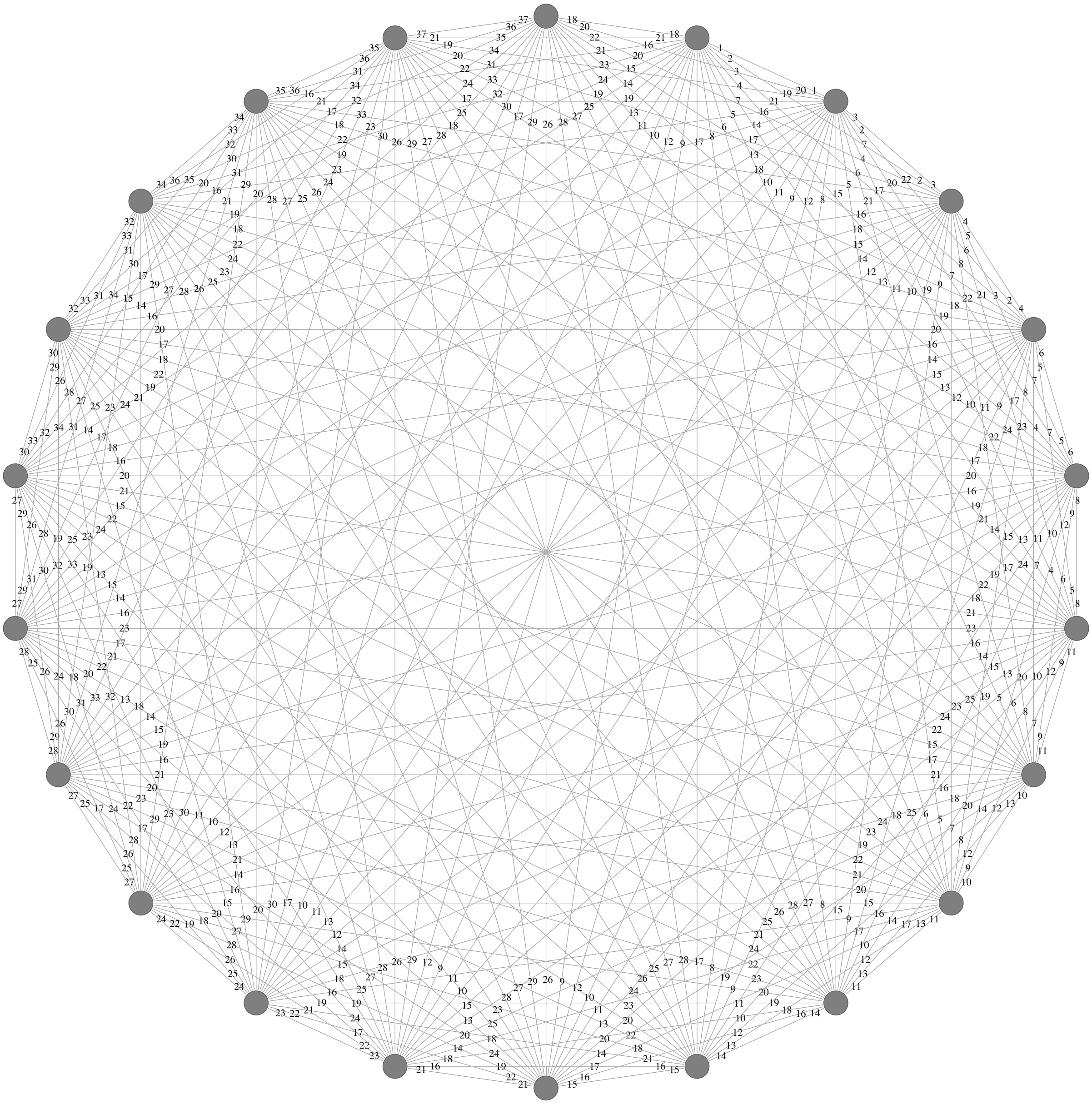

Fig. 6. Interval 37-coloring of K_{22} with a shift vector (1, 2, 1, 3, 1, 1, 3, 1, 2, 1).

Table 2
Bounds on $W(K_{2n})$: The first row lists the lower bounds from Theorem 11, the second row lists the known exact values and the third row lists the upper bounds from Theorem 10.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$W(K_{2n}) \geq$	1	4	7	11	14	18	21	26	29	33	37	41	42	46	52	57	56	64
$W(K_{2n}) =$	1	4	7	11	14	18	21	26	29	33	37	41				57		
$W(K_{2n}) \leq$	1	4	7	11	14	18	22	26	29	33	37	41	45	49	53	57	61	65

Proof. To prove the bound we take the bound from Corollary 2, set the exact values of $W(K_{2p_i})$ for the first five prime numbers and use Theorem 8 to bound $W(K_{2p_i})$ for $i \geq 6$, taking into account that all prime numbers except 2 are odd. $\square$

Table 2 lists obtained lower and upper bounds on $W(K_{2n})$ and all known exact values for $n \leq 18$.

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