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Sufficient conditions for Hamiltonian cycles in bipartite digraphs

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ABSTRACT

We prove two sufficient conditions for Hamiltonian cycles in balanced bipartite digraphs. Let D be a strongly connected balanced bipartite digraph of order $2a$. Then:

(i) *If $a \geq 4$ and $max\{d(x), d(y)\} \geq 2a-1$ for every pair of vertices $\{x, y\}$ with a common out-neighbour, then either D is Hamiltonian or D is isomorphic to a certain digraph of order eight which we specify.*

(ii) *If $a \geq 4$ and $d(x)+d(y) \geq 4a-3$ for every pair of vertices $\{x, y\}$ with a common out-neighbour, then D is Hamiltonian.*

The first result improves a theorem of Wang and the second result, in particular, establishes a conjecture due to Bang-Jensen, Gutin and Li for strongly connected balanced bipartite digraphs of orders at least eight.

1. Introduction

We consider directed graphs (digraphs) in the sense of [7]. Every cycle and path is assumed simple and directed. A digraph D is *Hamiltonian* if it contains a cycle passing through all the vertices of D. There are many conditions that guarantee that a digraph is Hamiltonian (see, e.g., [7,10,14,20,21,26]). Let us recall the following well-known degree conditions (Theorems 1.1–1.4).

Theorem 1.1 (*Nash-Williams [24]*). *Let D be a digraph of order $n \geq 3$ such that for every vertex x, $d^+(x) \geq n/2$ and $d^-(x) \geq n/2$, then D is Hamiltonian.*

Theorem 1.2 (*Ghouila-Houri [16]*). *Let D be a strongly connected digraph of order $n \geq 3$. If $d(x) \geq n$ for all vertices $x \in V(D)$, then D is Hamiltonian.*

Theorem 1.3 (*Woodall [28]*). *Let D be a digraph of order $n \geq 3$. If $d^+(x) + d^-(y) \geq n$ for all pairs of vertices x and y such that there is no arc from x to y, then D is Hamiltonian.*

Theorem 1.4 (*Meyniel [23]*). *Let D be a strongly connected digraph of order $n \geq 2$. If $d(x) + d(y) \geq 2n-1$ for all pairs of non-adjacent vertices in D, then D is Hamiltonian.*

It is easy to see that Meyniel's theorem is a generalization of Nash-Williams', Ghouila-Houri's and Woodall's theorems. A short proof of Theorem 1.4 can be found in [12].

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For bipartite digraphs, an analogue of Nash-Williams' theorem was given by Amar and Manoussakis in [5]. An analogue of Woodall's theorem was given by Manoussakis and Millis in [22], and strengthened by Adamus and Adamus [3]. The results analogous to the above-mentioned theorems of Ghouila-Houri and Meyniel for bipartite digraphs were given by Adamus, Adamus and Yeo [4].

Theorem 1.5 (*Adamus, Adamus, Yeo [4]*)**.** *Let D be a balanced bipartite digraph of order $2a$, where $a \geq 2$. Then D is Hamiltonian provided one of the following holds:*
(a) *$d(u) + d(v) \geq 3a + 1$ for every pair of non-adjacent distinct vertices u and v of D;*
(b) *D is strongly connected and $d(u) + d(v) \geq 3a$ for every pair of non-adjacent distinct vertices u and v of D;*
(c) *the minimal degree of D is at least $(3a + 1)/2$;*
(d) *D is strongly connected and the minimal degree of D is at least $3a/2$.*

Some sufficient conditions for the existence of Hamiltonian cycles in a bipartite tournament are described in the survey paper [18] by Gutin. A characterization for hamiltonicity for semicomplete bipartite digraphs was obtained independently by Gutin [17] and Häggkvist and Manoussakis [19].

Notice that each of Theorems 1.1–1.4 imposes a degree condition on all pairs of non-adjacent vertices (or on all vertices). In the following theorems a degree condition requires only for some pairs of non-adjacent vertices.

Theorem 1.6 (*Bang-Jensen, Gutin, Li [8]*)**.** *Let D be a strongly connected digraph of order $n \geq 2$. Suppose that $\min\{d(x), d(y)\} \geq n - 1$ and $d(x) + d(y) \geq 2n - 1$ for every pair of non-adjacent vertices x, y with a common in-neighbour. Then D is Hamiltonian.*

Theorem 1.7 (*Bang-Jensen, Guo, Yeo [6]*)**.** *Let D be a strongly connected digraph of order $n \geq 2$. Suppose that $\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geq n - 1$ and $d(x) + d(y) \geq 2n - 1$ for every pair of non-adjacent vertices x, y with a common in-neighbour or a common out-neighbour. Then D is Hamiltonian.*

An analogue of Theorem 1.6 for bipartite balanced digraphs was given by Wang [27].

Theorem 1.8 (*Wang [27]*)**.** *Let D be a strongly connected balanced bipartite digraph of order $2a$, where $a \geq 1$. Suppose that, for every pair of vertices $\{x, y\}$ with a common out-neighbour, either $d(x) \geq 2a - 1$ and $d(y) \geq a + 1$ or $d(y) \geq 2a - 1$ and $d(x) \geq a + 1$. Then D is Hamiltonian.*

In [8], Bang-Jensen, Gutin and Li raised the following two conjectures.

Conjecture 1 (*Bang-Jensen, Gutin, Li [8]*)**.** *Let D be a strongly connected digraph of order $n \geq 2$. Suppose that $d(x)+d(y) \geq 2n-1$ for every pair of non-adjacent vertices x, y with a common in-neighbour or a common out-neighbour. Then D is Hamiltonian.*

Conjecture 2 (*Bang-Jensen, Gutin, Li [8]*)**.** *Let D be a strongly connected digraph of order $n \geq 2$. Suppose that $d(x)+d(y) \geq 2n-1$ for every pair of non-adjacent vertices x, y with a common in-neighbour. Then D is Hamiltonian.*

Adamus [1] proved a bipartite analogue of Conjecture 1.

Theorem 1.9 (*Adamus [1]*)**.** *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 6$. If $d(x) + d(y) \geq 3a$ for every pair of vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.*

The above-mentioned result of Wang and Conjecture 2 were the main motivation for the present work.

Using some ideas and arguments of [27], in this paper we prove the following Theorems 1.10 and 1.11. For $a \geq 4$ Theorem 1.10 improves the theorem of Wang. Theorem 1.11, in particular, establishes Conjecture 2 for bipartite digraphs of orders at least 8 in a strong form.

Theorem 1.10. *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 8$. Suppose that $\max\{d(x), d(y)\} \geq 2a - 1$ for every pair of vertices x, y with a common out-neighbour. Then D is Hamiltonian unless D is isomorphic to the digraph $D(8)$ (for definition of $D(8)$, see Example 1).*

Theorem 1.11. *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 8$. Suppose that $d(x) + d(y) \geq 4a - 3$ for every pair of vertices x, y with a common out-neighbour. Then D is Hamiltonian.*

Observe that Theorem 1.11 and Wang's (when $a \geq 4$) theorem are immediate consequences of Theorem 1.10.

2. Terminology and notation

Terminology and notation not described below follow [7]. In this paper we consider finite digraphs without loops and multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and by $A(D)$, respectively. The *order* of D is the number of its vertices. For any $x, y \in V(D)$, we also write $x \to y$ if $xy \in A(D)$. If $xy \in A(D)$, then we say that x *dominates* y or y is an *out-neighbour* of x and x is an *in-neighbour* of y. The notation $x \leftrightarrow y$ denotes that $x \to y$ and $y \to x$ ($x \leftrightarrow y$ is called a *2-cycle*). Let x, y be distinct vertices in a digraph D. The pair $\{x, y\}$ called *dominating* if there is a vertex z in D such that $x \to z$ and $y \to z$.

For disjoint subsets A and B of $V(D)$ we define $A(A \to B)$ as the set $\{xy \in A(D) : x \in A, y \in B\}$; $A(A, B) = A(A \to B) \cup A(B \to A)$. If $x \in V(D)$ and $A = \{x\}$ we sometimes will write x instead of $\{x\}$. $A \to B$ means that every vertex of A dominates every vertex of B; $A \mapsto B$ means that $A \to B$ and there is no arc from B to A.

Let $N^+(x)$, $N^-(x)$ denote the set of out-neighbours, respectively the set of in-neighbours of a vertex x in a digraph D. If $A \subseteq V(D)$, then $N^+(x, A) = A \cap N^+(x)$, $N^-(x, A) = A \cap N^-(x)$ and $N^+(A) = \cup_{x \in A} N^+(x)$. The *out-degree* of x is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the *in-degree* of x. Similarly, $d^+(x, A) = |N^+(x, A)|$ and $d^-(x, A) = |N^-(x, A)|$. The *degree* of the vertex x in D is defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x, A) = d^+(x, A) + d^-(x, A)$). The subdigraph of D induced by a subset A of $V(D)$ is denoted by $D\langle A\rangle$.

For integers a and b, $a \le b$, let $[a, b]$ denote the set of all integers which are not less than a and are not greater than b.

The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \dots, x_m$ ($m \ge 2$) and the arcs $x_i x_{i+1}$, $i \in [1, m-1]$ (respectively, $x_i x_{i+1}$, $i \in [1, m-1]$, and $x_m x_1$), is denoted by $x_1 x_2 \cdots x_m$ (respectively, $x_1 x_2 \cdots x_m x_1$). We say that $x_1 x_2 \cdots x_m$ is a path from x_1 to x_m or is an (x_1, x_m)*-path*. Given a vertex x of a directed path P or a directed cycle C, we denote by x^+ (respectively, by x^-) the successor (respectively, the predecessor) of x (on P or C), and in case of ambiguity, we precise P or C as a subscript (that is $x_P^+ \dots$).

A cycle that contains all the vertices of D is a *Hamiltonian cycle*. A digraph D is *Hamiltonian* if it contains a Hamiltonian cycle. If P is a path containing a subpath from x to y, then by $P[x, y]$ we denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x, y]$ denotes the subpath of C from x to y (possibly, $x = y$). A digraph D is *strongly connected* (or, just, *strong*) if there exists an (x, y)-path in D for every ordered pair of distinct vertices x, y of D.

Two distinct vertices x and y are *adjacent* if $xy \in A(D)$ or $yx \in A(D)$ (or both).

Let H be a non-trivial proper subset of $V(D)$. An (x, y)-path P is a H*-bypass* if $|V(P)| \ge 3$, $x \ne y$ and $V(P) \cap H = \{x, y\}$.

A *cycle factor* in D is a collection of vertex-disjoint cycles $C_1, C_2, \dots, C_l$ such that $V(C_1) \cup V(C_2) \cup \cdots \cup V(C_l) = V(D)$. A digraph D is *bipartite* if there exists a partition X, Y of $V(D)$ into two partite sets such that every arc of D has its end-vertices in different partite sets. It is called *balanced* if $|X| = |Y|$. A *matching* from X to Y is an independent set of arcs with origin in X and terminus in Y. (A set of arcs with no common end-vertices is called *independent*). If D is balanced, one says that such a matching is *perfect* if it consists of precisely $|X|$ arcs.

Definition 2.1. Let D be a balanced bipartite digraph of order $2a$, where $a \ge 2$. For any integer k, we will say that D satisfies condition B_k when $max\{d(x), d(y)\} \ge 2a - 2 + k$ for every dominating pair of vertices $\{x, y\}$.

The underlying undirected graph of a digraph D, denoted by $UG(D)$, it contains an edge xy if $x \to y$ or $y \to x$ (or both).

3. Examples

In this section we present some examples of balanced bipartite digraphs which we will use in the next sections to show that the conditions of our results (the lemmas and the theorems) are sharp in some situation.

Example 1. Let $D(8)$ be a bipartite digraph with partite sets $X = \{x_0, x_1, x_2, x_3\}$ and $Y = \{y_0, y_1, y_2, y_3\}$, and the arc set $A(D(8))$ contains exactly the following arcs $y_0x_1, y_1x_0, x_2y_3, x_3y_2$ and all the arcs of the following 2-cycles: $x_i \leftrightarrow y_i$, $i \in [0, 3]$, $y_0 \leftrightarrow x_2$, $y_0 \leftrightarrow x_3$, $y_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow x_3$.

It is easy to see that

$$d(x_2) = d(x_3) = d(y_0) = d(y_1) = 7 \quad \text{and} \quad d(x_0) = d(x_1) = d(y_2) = d(y_3) = 3,$$

and the dominating pairs in $D(8)$ are: $\{y_0, y_1\}$, $\{y_0, y_2\}$, $\{y_0, y_3\}$, $\{y_1, y_2\}$, $\{y_1, y_3\}$, $\{x_0, x_2\}$, $\{x_0, x_3\}$, $\{x_1, x_2\}$, $\{x_1, x_3\}$ and $\{x_2, x_3\}$. Note that every dominating pair satisfies condition B_1. Since $x_0y_0x_3y_2x_2\,y_1x_0$ is a cycle in $D(8)$, it is not difficult to check that $D(8)$ is strong.

Observe that $D(8)$ is not Hamiltonian. Indeed, if C is a Hamiltonian cycle in $D(8)$, then C would contain the arcs x_1y_1 and x_0y_0. Therefore, C would contain the path $x_1y_1x_0y_0$ or the path $x_0y_0x_1y_1$, which is impossible since $N^-(x_0) = N^-(x_1) = \{y_0, y_1\}$.

Notice that the digraph $D(8)$ does not satisfy the conditions of Wang's theorem.

Example 2. Let $D(6)$ be a bipartite digraph with partite sets $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$, and arc set $A(D(6)) = \{x_iy_i, y_ix_i : i \in [1, 3]\} \cup \{x_1y_2, x_2y_1, x_1y_3, y_3x_1, x_2y_3, y_3x_2\}$.

Notice that $d(x_1) = d(x_2) = 5$, $d(y_1) = d(y_2) = 3$, $d(x_3) = 2$ and $d(y_3) = 6$. The dominating pairs in $D(6)$ are the following pairs $\{x_1, x_2\}$, $\{x_1, x_3\}$, $\{x_2, x_3\}$, $\{y_1, y_3\}$ and $\{y_2, y_3\}$ ($\{y_1, y_2\}$ is not a dominating pair). It is easy to check that $D(6)$ is strong and satisfies condition B_1, but $UG(D(6))$ is not 2-connected.

Example 3. Let $H(6)$ be a bipartite digraph with partite sets $X = \{x, y, z\}$ and $Y = \{u, v, w\}$, and arc set $A(H(6)) = \{xu, ux, vx, wx, yu, vy, uz, vz, zv, zw\}$.

Observe that $xuzwx$ is a cycle of length 4 in $H(6)$. The digraph $H(6)$ is strong, $d(x) = d(u) = d(v) = 4$ and the dominating pairs in $H(6)$ are the following pairs $\{x, y\}$, $\{u, v\}$, $\{u, w\}$ and $\{v, w\}$. Notice that $H(6)$ satisfies condition B_0, but contains no perfect matching from X to Y since $N^+(\{x, y\}) = \{u\}$. In particular, $H(6)$ is not Hamiltonian.

Example 4. Let D be a balanced bipartite digraph of order $2a \geq 8$ with partite sets $X = A \cup B \cup \{z\}$ and $Y = C \cup \{u, v\}$, where the subsets A and B are non-empty, $A \cap B = \emptyset$, $z \notin A \cup B$ and $u, v \notin C$. Let D satisfy the following conditions:

(i) the induced subdigraph $D\langle A \cup B \cup C \cup \{z\}\rangle$ is a complete bipartite digraph with partite sets $A \cup B \cup \{z\}$ and C;

(ii) $z \to u$ and $z \leftrightarrow v$;

(iii) $N^+(u) = A$, $N^+(v) = B \cup \{z\}$; and D contains no other arcs.

It is not difficult to check that D is strong, $d(x) = 2a - 3$ for all $x \in A \cup B$, $d(y) = 2a$ for all $y \in C$, $d(z) = 2a - 1$, $d(u) = |A| + 1$ and $d(v) = |B| + 2$. It is easy to check that $max\{d(b), d(c)\} \geq 2a - 3$ for every dominating pair of vertices $\{b, c\}$ (i.e, D satisfies condition B_{-1}). Since $N^+(A \cup B) = C$ and $a - 1 = |A \cup B| > |C| = a - 2$, by Köning–Hall theorem D contains no perfect matching from X to Y.

Example 5. Let H be the complete bipartite digraph of order $2a - 2 \geq 6$ with partite sets $X = \{x_1, x_2, \dots, x_{a-1}\}$ and $Y = \{y_1, y_2, \dots, y_{a-1}\}$. Let D be the digraph obtained from the digraph H by adding two new vertices x_0, y_0 and the following arcs x_0y_0, y_0x_0, x_0y_1, y_1x_0.

Clearly D is strongly connected and satisfies condition B_0, but $UG(D)$ is not 2-connected.

Example 6. Let $F(6)$ be the bipartite digraph with partite sets $X = \{x_0, x_1, x_2\}$ and $Y = \{y_0, y_1, y_2\}$, and arc set $A(F(6)) = \{x_iy_i, y_ix_i : i \in [0, 2]\} \cup \{y_0x_1, y_1x_2, x_0y_1, x_0y_2, y_1x_0, x_1y_2, y_2x_1\}$.

It is not difficult to check that $F(6)$ is strong and satisfies condition B_1, but $F(6)$ is not Hamiltonian.

4. Preliminaries

Bypass lemma (*Lemma 3.17, Bondy [11]*)**.** *Let D be a strong non-separable (i.e., $UG(D)$ is 2-connected) digraph, and let H be a non-trivial proper subdigraph of D. Then D contains a H-bypass.*

Remark. One can prove Bypass Lemma using the proof of Theorem 5.4.2 [7].

Now we will prove a series of lemmas.

Lemma 4.1. *Let D be a strong balanced bipartite digraph of order $2a \geq 8$ with partite sets X and Y. If D satisfies condition B_1, then the following statements hold:*

(i) *the underlying undirected graph $UG(D)$ is 2-connected;*

(ii) *if C is a cycle of length m, $2 \leq m \leq 2a - 2$, then D contains a C-bypass.*

Proof of Lemma 4.1. (i) Suppose, on the contrary, that D is strong and satisfies condition B_1 but $UG(D)$ is not 2-connected. Then $V(D) = E \cup F \cup \{u\}$, where E and F are non-empty subsets, $E \cap F = \emptyset$, $u \notin E \cup F$ and there is no arc between E and F. Since D is strong, it follows that there are vertices $x \in E$ and $y \in F$ such that $\{x, y\} \to u$, i.e., $\{x, y\}$ is a dominating pair. By condition B_1, $max\{d(x), d(y)\} \geq 2a - 1$. Without loss of generality, we assume that $x, y \in X$ and $d(x) \geq 2a - 1$. Then $u \in Y$. From $d(x) \geq 2a - 1$ and $A(E, F) = \emptyset$ it follows that $|E \cap Y| = a - 1$, i.e., $Y \cap F = \emptyset$. Since $a \geq 4$, there exist two distinct vertices in $Y \cap E$, say y_1, y_2, such that $\{y_1, y_2\} \to x$, i.e., $\{y_1, y_2\}$ is a dominating pair. Since $d(y, \{y_1, y_2\}) = 0$, we have $max\{d(y_1), d(y_2)\} \leq 2a - 2$, which contradicts condition B_1. This proves that $UG(D)$ is 2-connected.

(ii) The second claim of the lemma is an immediate consequence of the first claim and Bypass Lemma. Lemma 4.1 is proved. □

The digraph $D(6)$ (Example 2) shows that the bound on order of D in Lemma 4.1 is sharp.

The digraph of Example 5 shows that for any $a \geq 4$, if in Lemma 4.1 we replace condition B_1 with B_0, then the lemma is not true.

Using arguments similar to those of Lemma 4.1, we can easily prove the following lemma:

Lemma 4.2. *Let D be a strong balanced bipartite digraph of order $2a \geq 4$, with partite sets X and Y. If $d(x) + d(y) \geq 2a + 3$ for every dominating pair of vertices $\{x, y\}$, then*

(i) *the underlying undirected graph $UG(D)$ is 2-connected; and*

(ii) *if C is a cycle of length m, $2 \leq m \leq 2a - 2$, then D contains a C-bypass.*

Note that Lemma 4.2 is not needed for the proof of Theorem 1.10.

Lemma 4.3. *Let D be a strong balanced bipartite digraph of order $2a \geq 8$ with partite sets X and Y. If D satisfies condition B_0, then D contains a perfect matching from X to Y and a perfect matching from Y to X. Moreover, D contains a cycle factor.*

Proof of Lemma 4.3. Let D be a digraph satisfying the conditions of the lemma. By the well-known Köning–Hall theorem (see, e.g., [9]) to show that D contains a perfect matching from X to Y, it suffices to show that $|N^+(S)| \geq |S|$ for every set $S \subseteq X$. Let $S \subseteq X$. If $|S| = 1$ or $|S| = a$, then $|N^+(S)| \geq |S|$ since D is strong. Assume that $2 \leq |S| \leq a - 1$. We claim that $|N^+(S)| \geq |S|$. Suppose, that this is not the case, i.e., there exists S such that $|N^+(S)| \leq |S| - 1 \leq a - 2$. From this and strongly connectedness of D it follows that there are two vertices $x, y \in S$ and a vertex $z \in N^+(S)$ such that $\{x, y\} \to z$, i.e., $\{x, y\}$ is a dominating pair. Hence, by condition B_0, $max\{d(x), d(y)\} \geq 2a-2$. Without loss of generality, we assume that $d(x) \geq 2a-2$. It is easy to see that

$$2a-2 \leq d(x) \leq 2|N^+(S)| + a - |N^+(S)| = a + |N^+(S)|.$$

Therefore, $|N^+(S)| \geq a-2$. Thus, $|N^+(S)| = a-2$ and $|S| = a-1$ since $|N^+(S)| \leq a-2$. Now it is easy to see that $d(x) = 2a-2$, and hence, $\{u, v\} \to x$, where $\{u, v\} = Y \setminus N^+(S)$. By condition B_0, $max\{d(u), d(v)\} \geq 2a-2$. Without loss of generality, we assume that $d(u) \geq 2a-2$. On the other hand,

$$2a-2 \leq d(u) \leq |S| + 2(a-|S|) = 2a - |S|,$$

which implies that $|S| \leq 2$. Therefore, $a \leq 3$ since $|S| = a-1 \leq 2$. This contradicts that $a \geq 4$.

Thus, for any $S \subseteq X$ we have shown that $|N^+(S)| \geq |S|$. By the Köning–Hall theorem there exists a perfect matching from X to Y. The proof for a perfect matching in the opposite direction is analogous. Ore in [25] (Section 8.6) has shown that a balanced bipartite digraph D with partite sets X and Y has a cycle factor if and only if D contains a perfect matching from X to Y and a perfect matching from Y to X. Therefore, D contains a cycle factor. Lemma 4.3 is proved. □

The digraph $H(6)$ (Example 3) shows that the bound on order of D is sharp in Lemma 4.3.

The digraph D of Example 4 shows that if in Lemma 4.3 we replace condition B_0 with condition B_{-1}, then the lemma is not true.

Lemma 4.4. *Let D be a strong balanced bipartite digraph of order $2a \geq 8$ with partite sets X and Y. Suppose that D is not a directed cycle and satisfies condition B_0, i.e., $max\{d(x), d(y)\} \geq 2a-2$ for every dominating pair of vertices $\{x, y\}$. Then D contains a non-Hamiltonian cycle of length at least four.*

Proof of Lemma 4.4. Let D be a digraph satisfying the conditions of the lemma. If D is Hamiltonian, then it is not difficult to show that D contains a non-Hamiltonian cycle of length at least 4. So we suppose, from now on, that D is not Hamiltonian and contains no cycle of length at least 4. By Lemma 4.3, D contains a cycle factor. Let $C_1, C_2, \ldots, C_t$ be a minimal cycle factor of D (i.e., t is as small as possible). Then the length of every C_i is equal to two and $t = a$. Let $C_i = x_i y_i x_i$, where $x_i \in X$ and $y_i \in Y$. Since D is strong and is not a directed cycle, there exists a vertex such that its in-degree at least two, which means that there exists a dominating pair of vertices, say u and v. By condition B_0, $max\{d(u), d(v)\} \geq 2a-2$. Without loss of generality, we assume that $u, v \in X$, $u = x_1$ and

$$d(x_1) \geq 2a-2. \tag{1}$$

Since $a \geq 4$ and (1), there exists a vertex in $Y \setminus \{y_1\}$, say y_2, such that $x_1 \leftrightarrow y_2$. It is easy to see that y_1 and x_2 are not adjacent, for otherwise D would contain a cycle of length 4. We have that $\{y_1, y_2\} \to x_1$, i.e., $\{y_1, y_2\}$ is a dominating pair. Therefore, by condition B_0,

$$max\{d(y_1), d(y_2)\} \geq 2a-2. \tag{2}$$

If $d(y_1) \geq 2a-2$, then y_1 and every vertex x_i other than x_2 form a 2-cycle, since y_1 and x_2 are non-adjacent. This implies that D contains a 4-cycle, since x_1 is adjacent to every vertex of Y, maybe except one. We may therefore assume that $d(y_1) \leq 2a-3$. Then, by (2), $d(y_2) \geq 2a-2$. Consider the following two possible cases.

Case 1. The vertex y_2 and some vertex in $X \setminus \{x_1, x_2\}$, say x_3, form a 2-cycle, i.e., $y_2 \leftrightarrow x_3$.

Then it is not difficult to see that $max\{d(x_3), d(x_2)\} \geq 2a-2$ since $\{x_2, x_3\}$ is a dominating pair. Since D contains no cycle of length 4, it is not difficult to check that

$$d(x_1, \{y_3\}) = d(x_2, \{y_1, y_3\}) = d(x_3, \{y_1\}) = 0.$$

These imply that $d(x_2) \leq 2a-4$, $d(x_3) \geq 2a-2$, $x_3 \leftrightarrow y_4$, and $x_1 \leftrightarrow y_4$ because of $d(x_1) \geq 2a-2$. Therefore, $x_1 y_4 x_3 y_2 x_1$ is a cycle of length 4, which contradicts our supposition that D contains no cycle of length at least 4.

Case 2. $d(y_2, \{x_i\}) \leq 1$ for all $x_i \notin \{x_1, x_2\}$.

In this case from $d(y_2) \geq 2a-2$ it follows that $a = 4$ and $d(y_2, \{x_i\}) = 1$ if $i \in \{3, 4\}$.

First consider the case $d^+(y_2, \{x_3, x_4\}) \geq 1$. Without loss of generality, we may assume that $y_2 \mapsto x_3$. Using the supposition that D contains no cycle of length at least 4, it is not difficult to show that $d^+(y_3, \{x_1, x_2\}) = 0$, $x_3 y_1 \notin A(D)$. Therefore,

$$A(\{x_3, y_3\} \to \{x_1, y_1, x_2, y_2\}) = \emptyset. \tag{3}$$

If $y_2 \to x_4$, by an argument similar to that in the proof of (3), we obtain that $A(\{x_4, y_4\} \to \{x_1, y_1, x_2, y_2\}) = \emptyset$, which together with (3) contradicts that D is strong. We may therefore assume that $y_2 x_4 \notin A(D)$. Then $x_4 \mapsto y_2$, since $d(y_2, \{x_4\}) = 1$. From $d(x_2, \{y_1\}) = d^-(x_2, \{y_3\}) = 0$, we have that $d(x_2) \leq 2a-3$. From this, $\{x_2, x_4\} \to y_2$ and condition B_0 it follows that $d(x_4) \geq 2a-2$. On the other hand, using the supposition that D contains no cycle of length at least 4, it is easy to check that $d^-(x_4, \{y_1, y_3\}) = 0$. This together with $d(x_4, \{y_2\}) = 1$ gives $d(x_4) \leq 2a-3$, which is a contradiction.

Now consider the case $d^+(y_2, \{x_3, x_4\}) = 0$. Then $\{x_3, x_4\} \mapsto y_2$ because of $d(y_2, \{x_3\}) = d(y_2, \{x_4\}) = 1$. Since D contains no cycle of length at least 4, it is easy to check that $A(\{x_1, x_2\} \to \{y_3, y_4\}) = \emptyset$ and $d^+(y_1, \{x_3, x_4\}) = 0$. In consequence, we have $A(\{x_1, y_1, x_2, y_2\} \to \{x_3, y_3, x_4, y_4\}) = \emptyset$, which contradicts that D is strong. Lemma 4.4 is proved. □

Let C_6^* (respectively, P^*) be the digraph obtained from the undirected cycle of length 6 (respectively, from undirected path of length 5) by replacing every edge xy with the pair xy, yx of arcs.

Observe that the digraph C_6^* (and P^*) satisfies the conditions of Lemma 4.4, but has no cycle of length 4.

5. Proof of the main result

Proof of Theorem 1.10. Let D be a digraph satisfying the conditions of the theorem. For a proof by contradiction, suppose that D is not Hamiltonian. In particular, D is not isomorphic to the directed cycle of length $2a$. Let $C := x_0y_0x_1y_1 \dots x_{m-1}y_{m-1}x_0$ be a longest cycle in D, where $x_i \in X$ and $y_i \in Y$ for all $i \in [0, m-1]$ (all subscripts of x_i and y_i are taken modulo m, i.e., $x_{m+i} = x_i$ and $y_{m+i} = y_i$ for all $i \in [0, m-1]$). By Lemma 4.4, D contains a cycle of length at least 4, i.e., $m \geq 2$. By Lemma 4.1(ii), D has a C-bypass. Let $P := xu_1u_2 \dots u_sy$ be a C-bypass ($s \geq 1$). The length of the path $C[x, y]$ is the gap of P with respect to C. Suppose also that the gap of P is minimum among the gaps of all C-bypass. Since C is a longest cycle in D, the length of $C[x, y]$ is greater than or equal to $s + 1$.

Firstly we prove that $s = 1$. Suppose, on the contrary, that is $s \geq 2$. Since C is a longest cycle in D and P has the minimum gap among the gaps of all C-bypass, the vertex y_C^- (respectively, u_s) and every vertex of $P[u_1, u_s]$ (respectively, $C[x_C^+, y_C^-]$) are not adjacent. Hence,

$$d(u_s) \leq 2a - 2 \quad \text{and} \quad d(y_C^-) \leq 2a - 2$$

since each of $P[u_1, u_s]$ and $C[x_C^+, y_C^-]$ contains at least one vertex from each partite set. On the other hand, since $\{u_s, y_C^-\}$ is a dominating pair, by condition B_1, we have

$$2a - 1 \leq max\{d(u_s), d(y_C^-)\} \leq 2a - 2,$$

a contradiction. We have thus shown that $s = 1$.

Since $s = 1$ and D is a bipartite digraph, it follows that x and y belong to the same partite set and the length of $C[x, y]$ must be even. Now assume, without loss of generality, that $x = x_0$, $y = x_r$ and $v := u_1$. Let $C' := V(C[y_0, y_{r-1}])$ and $R := V(D) \setminus V(C)$. We now consider the cases when $r \geq 2$ and when $r = 1$ separately.

Case 1. $r \geq 2$.

Let x be an arbitrary vertex in $X \cap R$. Since C is a longest cycle in D, it is easy to see that

$$d^+(y_{r-1}, \{x\}) + d^+(x, \{v\}) \leq 1 \quad \text{and} \quad d^+(v, \{x\}) + d^+(x, \{y_0\}) \leq 1. \tag{4}$$

Note that $\{v, y_{r-1}\}$ is a dominating pair. Observe that v and every vertex of $C'$are not adjacent since C-bypass P has the minimum gap among the gaps of all C-bypass. Therefore,

$$d(v) \leq 2a - 2 \quad \text{and} \quad d(x_i) \leq 2a - 2, \tag{5}$$

where x_i is an arbitrary vertex in $X \cap C'$. Since $\{v, y_{r-1}\}$ is a dominating pair, using condition B_1 and the first inequality of (5), we obtain

$$d(y_{r-1}) \geq 2a - 1, \tag{6}$$

which in turn implies that

(i) the vertex y_{r-1} and every vertex of X are adjacent.

In particular, **(i)** implies that y_{r-1} and x are adjacent, i.e., $x \to y_{r-1}$ or $y_{r-1} \to x$. If $x \to y_{r-1}$, then $d^-(x, \{v, y_0\}) = 0$ because of gap minimality. Hence, $d(x) \leq 2a-2$. This together with $d(x_{r-1}) \leq 2a-2$ (by the second inequality of (5)) gives a contradiction since $\{x, x_{r-1}\}$ is a dominating pair. We may therefore assume that $xy_{r-1} \notin A(D)$. Then $y_{r-1} \mapsto x$. By the arbitrariness of x, we may assume that $y_{r-1} \mapsto X \cap R$. This together with $d(y_{r-1}) \geq 2a - 1$ (by (6)) implies that $|R| = 2$, i.e., the cycle C has length equal to $2a - 2$, and

(ii) the vertex y_{r-1} and every vertex of $X \cap V(C)$ form a 2-cycle. In particular, $\{x_0, x_1\} \to y_{r-1}$, $y_{r-1} \to \{x_0, x_1\}$, and $d(x_0) \geq 2a - 1$ since, by (5), $d(x_1) \leq 2a - 2$.

By **(ii)**, any two distinct vertices of $X \cap V(C)$ form a dominating pair. Therefore, every vertex of $X \cap V(C)$, except for at most one vertex, has degree at least $2a - 1$. This together with the second inequality of (5) implies that $r = 2$ and

$$d(x_i) \geq 2a - 1, \quad \text{for all} \quad x_i \in \{x_0, x_1, \dots, x_{m-1}\} \setminus \{x_1\}, \tag{7}$$

which in turn implies that

(iii) the vertex v and every vertex $x_i \in \{x_0, x_1, \dots, x_{m-1}\} \setminus \{x_1\}$ are adjacent.

It is not difficult to see that

$$\text{if} \quad x \to y_0, \quad \text{then} \quad d^+(y_j, \{x_1\}) + d^+(y_0, \{x_{j+1}\}) \leq 1 \quad \text{for all} \quad j \in [2, m-1]. \tag{8}$$

Indeed, if this is not the case, then $x \to y_0$ and there exists $i \in [2, m-1]$ such that $y_i \to x_1$ and $y_0 \to x_{i+1}$. Then, since $y_1 \to x$, we have that $vx_2 \dots y_ix_1y_1xy_0x_{i+1} \dots x_0v$ is a Hamiltonian cycle, which is a contradiction.

Using the first inequality of (4) and $xy_1 \notin A(D)$ (by our assumption that $xy_{r-1} \notin A(D)$) we obtain that $d^+(x, \{v, y_1\}) = 0$. Therefore, since D is strong, it follows that there is a vertex y_l other than y_1, such that $x \to y_l$. Notice that $l \neq 2$, since P has the minimum gap among the gaps of all C-bypass. If $l \leq m-1$ and $x_l \to y_0$, then $vx_2 \ldots x_l y_0 x_1 y_1 x y_l \ldots x_0 v$ is a Hamiltonian cycle, a contradiction. Thus, we may assume that

$$\text{if} \quad 3 \leq l \leq m-1, \quad \text{then} \quad x_l y_0 \notin A(D). \tag{9}$$

Recall that $r = 2$ and $|R| = 2$, and consider the following three possible subcases.

Subcase 1.1. $v \to x_0$.

From the minimality of the gap $|C[x_0, x_r]| - 1$ of P and **(iii)** it follows that $v \mapsto \{x_2, x_3, \ldots, x_{m-1}\}$. This together with (7) implies that

(iv) every vertex x_i, other than x_0 and x_1, and every vertex y_j form a 2-cycle. In particular, for all $i \in [2, m-1]$, $x_i \leftrightarrow y_0$ and $y_i \to x_2 \to \{y_0, y_1\}$.

Now using (9) and **(iv)**, it is not difficult to see that

$$d^+(x, \{y_1, y_2, \ldots, y_{m-1}\}) = 0 \quad \text{and} \quad x \to y_0, \tag{10}$$

i.e., $l = 0$. From $y_1 \to x \to y_0$ and (4) it follows that v and x are not adjacent. Therefore, $d(x) \leq 2a-2$. This together with $d(x_1) \leq 2a-2$ (by (5)) and condition B_1 implies that $x_1 y_0 \notin A(D)$. Since $\{v, y_0\} \to x_2$ and $d(v) \leq 2a-2$ (by (5)), from condition B_1 it follows that $d(y_0) \geq 2a-1$. This together with $x_1 y_0 \notin A(D)$ gives $y_0 \to x_0$. Combining this with **(iv)** we obtain that $y_0 \to \{x_2, x_3, \ldots, x_{m-1}, x_0\}$. Therefore, from (8) it follows that $d^-(x_1, \{y_2, y_3, \ldots, y_{m-1}\}) = 0$. This together with (10) implies that $d(y_2) \leq 2a-2$. Now recall that $\{v, y_2\} \to x_2$, by **(iv)** (i.e., $\{v, y_2\}$ is a dominating pair), but $d(y_2) \leq 2a-2$ and $d(v) \leq 2a-2$, which contradicts condition B_1.

Subcase 1.2. $vx_0 \notin A(D)$ and $x_2 \to v$.

Then from the minimality of the gap $|C[x_0, x_2]| - 1$ and **(iii)** it follows that

$$\{x_3, x_4, \ldots, x_{m-1}, x_0\} \mapsto v.$$

This together with (7) implies that

(v) every vertex x_i, other than x_1 and x_2, and every vertex y_j form a 2-cycle, where $j \in [0, m-1]$. In particular, $y_j \to x_0$, and if $i \notin \{1, 2\}$, then $x_i \leftrightarrow y_0$.

From (9) and **(v)** it follows that in this subcase (10) also is true. From $x \to y_0 \to x_i$, where $i \notin \{1, 2\}$, and (8) it follows that $d^-(x_1, \{y_2, y_3, \ldots, y_{m-1}\}) = 0$. Using this and (10), we obtain

$$d(y_j) \leq 2a-2 \quad \text{for all} \quad j \in [2, m-1]. \tag{11}$$

By **(v)**, $y_j \to x_0$ for all y_j. Combining this with (11) we obtain that $m = 3$, i.e., the cycle C has length 6, in particular, $a = 4$. From $d(y_2) \leq 2a-2$ (by (11)), $\{y_0, y_2\} \to x_0$ (by **(v)**) and condition B_1 it follows that $d(y_0) \geq 2a-1$. Therefore, $y_0 \to x$ and $y_0 \leftrightarrow x_2$ since $x_1 y_0 \notin A(D)$. Using **(ii)**, i.e., the fact that the vertex y_1 forms a 2-cycle together with each vertex of $\{x_0, x_1, x_2\}$, it is easy to see that x_1 and y_2 are not adjacent (for otherwise, if $x_1 \to y_2$, then $x_1 y_2 x_0 v x_2 y_1 x y_0 x_1$ is a Hamiltonian cycle; if $y_2 \to x_1$, then $y_2 x_1 y_1 x y_0 x_0 v x_2 y_2$ is a Hamiltonian cycle). On the other hand, the vertices x and y_2 also are not adjacent, because of the minimality of the gap $|C[x_0, x_2]| - 1$. Therefore, $d(y_2, \{x, x_1\}) = 0$. Since $v \to x_2$, $d(v) = 3$ and $d(y_2) \leq 4$, using condition B_1 we obtain that $y_2 x_2 \notin A(D)$. We have thus shown that $a = 4$, D contains exactly the following 2-cycles and arcs: $v \leftrightarrow x_2, x_2 \leftrightarrow y_1, y_1 \leftrightarrow x_1, y_1 \leftrightarrow x_0, y_2 \leftrightarrow x_0, y_0 \leftrightarrow x_0, y_0 \leftrightarrow x, y_0 \leftrightarrow x_2, x_2 y_2, y_1 x, y_0 x_1$ and $x_0 v$.

Now it is not difficult to check that D is isomorphic to $D(8)$. (To check this, let now $X := \{x_0, x_1, x_2, x_3\}$ and $Y := \{y_0, y_1, y_2, y_3\}$, where $x_0 := x$, $x_1 := x_1$, $x_2 := x_0$, $x_3 := x_2$, $y_0 := y_0$, $y_1 := y_1$, $y_2 := y_2$ and $y_3 := v$). Subcase 1.2 is considered.

Subcase 1.3. $vx_0 \notin A(D)$ and $x_2 v \notin A(D)$.

Let t be the number of vertices in $C[x_3, x_{m-1}]$ each of which together with v forms a 2-cycle (recall that $v \in Y$). We will consider the subcases $t \geq 1$ and $t = 0$ separately.

Subcase 1.3.1. $t \geq 1$.

Then $m \geq 4$. Let $x_q \in C[x_3, x_{m-1}]$ be a vertex such that v and x_q form a 2-cycle, i.e., $v \leftrightarrow x_q$. From this, **(iii)** and the fact that C-bypass P has the minimum gap among the gaps of all C-bypass it follows that

$$v \mapsto \{x_2, x_3, \ldots, x_{q-1}\} \quad \text{and} \quad \{x_{q+1}, x_{q+2}, \ldots, x_{m-1}, x_0\} \mapsto v. \tag{12}$$

Hence, $t = 1$. Using (7) and (12) we conclude that

(vi) every vertex $x_i \in C[x_2, x_0] \setminus \{x_q\}$ and every vertex y_j form a 2-cycle. In particular, for every vertex y_j, $y_j \leftrightarrow x_2$, $\{v, y_j\} \to x_2$ (i.e., $\{v, y_j\}$ is a dominating pair) and $x_i \to y_0$ for all x_i other than x_1 and x_q.

From condition B_1, **(vi)** and $d(v) \leq 2a-2$ (by (5)) it follows that

$$d(y_j) \geq 2a-1 \quad \text{for all} \quad j \in [0, m-1]. \tag{13}$$

Using $x_i \leftrightarrow y_0$ (by **(vi)**), where $x_i \notin \{x_1, x_q\}$, and (9) we obtain that $l = q$ or $l = 0$. Recall that $x \to y_l$.

Let $l = q$, i.e., $x \to y_q$. By (9), $x_q y_0 \notin A(D)$. This together with $d(x_q) \geq 2a - 1$ (by (7)) implies that $y_0 \to x_q$. Since C-bypass P has the minimum gap among the gaps of all C-bypass, it follows that $y_{q-1}x \notin A(D)$ (for otherwise, the C-bypass $y_{q-1} \to x \to y_q$ has a gap equal to 2, which is a contradiction). Therefore, since $d(y_{q-1}) \geq 2a - 1$ (by (13)), $y_{q-1} \to x_1$. Now it is easy to see that $vx_2 \dots y_{q-1}x_1y_1xy_q \dots x_0y_0x_qv$ is a Hamiltonian cycle in D, which contradicts our initial supposition.

Let now $l \neq q$. Then $l = 0$, i.e.,

$$x \to y_0 \quad \text{and} \quad d^+(x, C[y_1, y_{m-1}]) = 0,$$

in particular, $xy_{m-1} \notin A(D)$. Because of gap minimality and $x \to y_0$, we have that $y_{m-1}x \notin A(D)$. Therefore, x and y_{m-1} are not adjacent which in turn implies that $d(y_{m-1}) \leq 2a - 2$. This contradicts (13), when $j = m - 1$.

Subcase 1.3.2. $t = 0$, i.e., there is no x_i, $i \in [0, m-1]$, such that $x_i \leftrightarrow v$.

From (7) it follows that

(vii) every vertex x_i other than x_1 and every vertex y_j form a 2-cycle. In particular, for every $i \neq 1$ and every $j \in [1, m-1]$ we have $x_i \leftrightarrow y_0$ and $y_j \leftrightarrow x_2$.

Now using (9) and $x_i \leftrightarrow y_0$, $i \neq 1$, we see that for this subcase (10) also is true. From $x \to y_0$, **(vii)** (i.e., $y_0 \to \{x_2, x_3, \dots, x_{m-1}, x_0\}$) and (8) we obtain that $d^-(x_1, \{y_2, y_3, \dots, y_{m-1}\}) = 0$. This together with the equality of (10) implies that

$$d(y_j) \leq 2a - 2 \quad \text{for all} \quad y_j \notin \{y_0, y_1\},$$

in particular, $d(y_2) \leq 2a - 2$. From **(vii)** we have that $y_2 \to x_2$. Hence, $\{v, y_2\} \to x_2$ (i.e., $\{v, y_2\}$ is a dominating pair). Now using (5), we see that $max\{d(v), d(y_2)\} \leq 2a - 2$, which contradicts condition B_1. This contradiction completes the discussion of Case 1.

Case 2. $r = 1$.

Note that $\{v, y_0\}$ is a dominating pair. By condition B_1, $max\{d(v), d(y_0)\} \geq 2a - 1$. Because of the symmetry between the vertices v and y_0, we can assume that $d(v) \geq 2a - 1$, which implies that

(viii) the vertex v and every vertex of X are adjacent.

Subcase 2.1. In subdigraph $D\langle R\rangle$ there exists a 2-cycle through v.

Let $u \in X \cap R$ and $v \leftrightarrow u$. In this subcase, since C is a longest cycle in D, it is easy to see that the following Claims 1 and 2 are true.

Claim 1. *If $x_i \to v$, $i \in [0, m-1]$, then $uy_i \notin A(D)$, and if $v \to x_i$, then $y_{i-1}u \notin A(D)$.*

Claim 2. *If $x_i \to v \to x_{i+1}$, $i \in [0, m-1]$, then u and y_i are not adjacent.*

We now prove the following claim:

Claim 3. *If $x_i \leftrightarrow v$, $i \in [0, m-1]$, then (a) $x_{i+1} \mapsto v$ and (b) $v \mapsto x_{i-1}$ are impossible.*

Proof of Claim 3. (a). Suppose, on the contrary, that for some $i \in [0, m-1]$ $x_i \leftrightarrow v$ and $x_{i+1} \mapsto v$. This and the fact that $d(v) \geq 2a - 1$ (by our assumption) imply that

$$v \leftrightarrow x \quad \text{for every} \quad x \in X \setminus \{x_{i+1}\}. \tag{14}$$

From (14) and Claim 2 it follows that

(ix) if $v \leftrightarrow z$, where $z \in X \cap R$, then z and every vertex of $(Y \cap V(C)) \setminus \{y_i\}$ are not adjacent. In particular, the following hold $d(z) \leq 2a - 2$ and $d(y_j) \leq 2a - 2$ for any y_j other than y_i.

If $|X \cap R| \geq 2$, then, by (14) and **(ix)**, there are two distinct vertices in $X \cap R$, say x and z, such that $x \leftrightarrow v$, $z \leftrightarrow v$ and $max\{d(x), d(z)\} \leq 2a - 2$, which contradicts condition B_1. We may therefore assume that $|X \cap R| = 1$. Then the cycle C has length $2a - 2$, i.e., $m = a - 1 \geq 3$. Since $u \leftrightarrow v$, $x_{i+1} \mapsto v$ and $d(u) \leq 2a - 2$ (by **(ix)**), from condition B_1 it follows that $d(x_{i+1}) \geq 2a - 1$. This together with $x_{i+1} \mapsto v$ implies that x_{i+1} and every vertex of $Y \setminus \{v\}$ form a 2-cycle, i.e., any two distinct vertices of $Y \cap V(C)$ form a dominating pair. On the other hand, since $m \geq 3$ and **(ix)**, there exist two distinct vertices in $(Y \cap V(C)) \setminus \{y_i\}$, say y_s and y_k, such that $max\{d(y_s), d(y_k)\} \leq 2a - 2$, which contradicts condition B_1. Claim 3(a) is proved.

(b). Suppose, on the contrary, that for some $i \in [0, m-1]$ $x_i \leftrightarrow v$ and $v \mapsto x_{i-1}$. Similar to (14), we obtain that

$$v \leftrightarrow x \quad \text{for any} \quad x \in X \setminus \{x_{i-1}\}. \tag{15}$$

This and Claim 2 imply

(x) every vertex $x \in X \cap R$ and every vertex of $y_j \in (Y \cap V(C)) \setminus \{y_{i-1}\}$ are not adjacent. In particular, $d(x) \le 2a-2$ and $d(y_j) \le 2a-2$.

If $|X \cap R| \ge 2$, then, by (15) and **(x)**, there exist two distinct vertices in $X \cap R$, say x and z, such that $\{x, z\} \to v$ and $max\{d(x), d(z)\} \le 2a-2$, which contradicts condition B_1. Hence we may assume that $|X \cap R| = 1$. Then the cycle C has length $2a-2$, i.e., $m = a-1 \ge 3$. Since $x_i \leftrightarrow v$ and $v \leftrightarrow u$, from condition B_1 and the first inequality of **(x)** (when $x = u$) it follows that $d(x_i) \ge 2a-1$. Therefore, x_i and every vertex of $Y \cap V(C)$, maybe except one, form a 2-cycle. Using this and the second inequality of **(x)**, we obtain that $m = 3$, $y \to x_i$ for some $y \in (Y \cap V(C)) \setminus \{y_{i-1}\}$ and $d(y_{i-1}) \ge 2a-1$. Since $y_{i-1}u \notin A(D)$, it follows that $u \to y_{i-1}$. Thus we have, $\{x_{i-1}, u\} \to y_{i-1}$ (i.e., $\{x_{i-1}, u\}$ is a dominating pair) and $d(x_{i-1}) \ge 2a-1$ because of $d(u) \le 2a-2$ by the first inequality of **(x)**. Then $d(x_{i-1}) \ge 2a-1$ and $v \mapsto x_{i-1}$ imply that x_{i-1} and every vertex of $Y \cap V(C)$ form a 2-cycle. Since $m = 3$, there exist two distinct vertices in $(Y \cap V(C)) \setminus \{y_{i-1}\}$, say y_s and y_k, such that $\{y_s, y_k\} \to x_{i-1}$. Because of the second inequality of **(x)**, we have $max\{d(y_s), d(y_k)\} \le 2a-2$, which contradicts condition B_1. Claim 3 is proved. □

Now we can finish the proof of Theorem 1.10 in Subcase 2.1.

By Claim 3 and $d(v) \ge 2a-1$ (by our assumption), the vertex v and every vertex of $X \cap V(C)$ form a 2-cycle. Therefore, by Claim 2, the vertex u and every vertex of $V(C)$ are not adjacent, which in turn implies that

$$d(u) \le 2a-2 \quad \text{and} \quad d(y_j) \le 2a-2 \quad \text{for all} \quad j \in [0, m-1]. \tag{16}$$

Using the facts that $\{x_i, u\} \to v$, $d(u) \le 2a-2$ and condition B_1, we obtain that $d(x_i) \ge 2a-1$ for all x_i. Since $d(y_j) \le 2a-2$ for all y_j, using condition B_1 we conclude that no two distinct vertices of $\{y_0, y_1, \dots, y_{m-1}\}$ form a dominating pair. In particular, $y_i x_i \notin A(D)$ for all y_i. This and the fact that $d(x_i) \ge 2a-1$ imply that x_i and every vertex of $Y \setminus \{y_i\}$ form a 2-cycle.

From this and (16) it follows that if $m \ge 3$, then $\{y_0, y_2\} \to x_1$, but $max\{d(y_0), d(y_2)\} \le 2a-2$, which is a contradiction. We may therefore assume that $m = 2$. Then $|R| \ge 4$ and there is a vertex $y \in (Y \cap R) \setminus \{v\}$ such that $x_0 \leftrightarrow y$ since $y_0x_0 \notin A(D)$ and $d(x_0) \ge 2a-1$. Therefore, $\{y_1, y\} \to x_0$, i.e., $\{y_1, y\}$ is a dominating pair. Since $d(y_1) \le 2a-2$ (by (16)), condition B_1 implies that $d(y) \ge 2a-1$. Therefore, $y \to u$ or $u \to y$. Now using the facts that $x_0 \leftrightarrow y$, $x_0 \leftrightarrow v$ and $x_1 \leftrightarrow v$, it is not difficult to show that (in both cases) D contains a cycle of length $2m+2 = 6$, which contradicts that C is a longest cycle in D. The discussion of Subcase 2.1 is completed.

Subcase 2.2. In subdigraph $D\langle R\rangle$ there is no 2-cycle through the vertices v.

In this subcase from $d(v) \ge 2a-1$ it follows that $|R| = 2$, $u \mapsto v$ or $v \mapsto u$, where $\{u\} = X \cap R$, and the vertex v and every vertex of $X \setminus \{u\}$ form a 2-cycle. Since D is strong, it follows that if $u \mapsto v$ (respectively, $v \mapsto u$), then there exists a vertex y_i such that $y_i \to u$ (respectively, $u \to y_i$) and hence, $y_i u v x_{i+1} \dots x_i y_i$ (respectively, $x_i v u y_i \dots x_i$) is a Hamiltonian cycle, a contradiction. This contradiction completes the proof of the theorem. □

Theorem 1.10 is best possible in the following sense:

The digraph $F(6)$ (Example 6) and its converse digraph show that the bound on order of D in Theorem 1.10 is sharp.

The digraph D of Example 5 shows that if in Theorem 1.10 we replace condition B_1 with condition B_0, then the theorem is not true.

Corollary 5.1 (*Wang [27]*). *Let D be a strongly connected balanced bipartite digraph of order $2a$, where $a \ge 4$. Suppose that, for every dominating pair of vertices $\{x, y\}$, either $d(x) \ge 2a-1$ and $d(y) \ge a+1$ or $d(y) \ge 2a-1$ and $d(x) \ge a+1$. Then D is Hamiltonian.*

6. Concluding remarks

A balanced bipartite digraph of order $2a$ is *even pancyclic* if it contains cycles of every length $2k$, $2 \le k \le a$. Bondy suggested (see [13] by Chvátal) the following metaconjecture:

Metaconjecture. *Almost any non-trivial condition of a graph (digraph) which implies that the graph (as digraph) is Hamiltonian also implies that the graph (digraph) is pancyclic. (There may be a "simple" family of exceptional graphs (digraphs)).*

There are various sufficient conditions for a digraph (undirected graph) to be Hamiltonian are also sufficient for the digraph (undirected graph) to be pancyclic. Motivated by these, it is natural to set the following problem:

Problem. Characterize those balanced bipartite digraphs which satisfy condition B_1 or the condition of Theorem 1.9 but are not even pancyclic.

We have shown the following theorem.

Theorem 6.1 (*Darbinyan [15]*). *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 8$ other than a directed cycle of length $2a$. If $\max\{d(x), d(y)\} \geq 2a-1$ for every dominating pair of vertices $\{x, y\}$, then either D is even pancyclic or D is isomorphic to the digraph $D(8)$ (for the definition of $D(8)$, see Example 1).*

Using Theorem 1.9, recently Adamus [2] proved that:

Theorem 6.2 (*Adamus [2]*). *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 6$. Suppose that D is not a directed cycle and $d(x) + d(y) \geq 3a$ for every pair of vertices x, y with a common in-neighbour or a common out-neighbour. Then D is even pancyclic.*

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