A Sufficient Condition for pre-Hamiltonian Cycles in Bipartite Digraphs

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ABSTRACT

Let D be a strongly connected balanced bipartite directed graph of order $2a \geq 10$ other than a directed cycle. Let x, y be distinct vertices in D. $\{x, y\}$ dominates a vertex z if $x \to z$ and $y \to z$; in this case, we call the pair $\{x, y\}$ dominating. In this paper we prove:

If $max\{d(x), d(y)\} \geq 2a - 2$ for every dominating pair of vertices $\{x, y\}$, then D contains cycles of all lengths $2, 4, \ldots, 2a - 2$ or D is isomorphic to a certain digraph of order ten which we specify.

Keywords

Digraphs, pre-Hamiltonian cycles, bipartite balanced digraphs, even pancyclic.

1. INTRODUCTION

We consider digraphs (directed graphs) in the sense of [4], and use standard graph theoretical terminology and notation (see Section 2 for details). A cycle passing through all the vertices of a digraph is called Hamiltonian. A digraph containing a Hamiltonian cycle is called a Hamiltonian digraph. A digraph D of order n is called pancyclic if it contains cycles of every lengths $3, 4, \ldots, n$. Various sufficient conditions for a digraph to be Hamiltonian have been given in terms of the vertex degree of the digraph. Here we recall some of them which are due to Ghouila-Houri [16], Nash-Williams [23], Woodall [27], Meyniel [22], Thomassen [25] and Darbinyan [11]. The Meyniel theorem is a generalization Nash-Williams', Ghouila-Houri's and Woodall's theorems.

Bondy suggested (see [9] by Chvátal) the following metaconjecture:

Metaconjecture. *Almost any non-trivial condition of a graph (digraph) which implies that the graph (digraph) is Hamiltonian also implies that the graph (digraph) is pancyclic. (There may be a "simple" family of exceptional graphs (digraphs)).*

In fact various sufficient conditions for a digraph to be Hamiltonian are also sufficient for the digraph to be pancyclic. Namely, in [20, 24, 10, 12] it was shown that if a digraph D satisfies one of the above mentioned conditions for hamiltonicity digraphs, then the digraph D also is pancyclic (unless some extremal cases which are characterized). For additional information on Hamiltonian and pancyclic digraphs, see, e.g., the book by Bang-Jensen and Gutin [4] and the surveys [7] by Bermond and Thomassen, [21] by Kühn and Ostus and [17] by Gutin.

Each of the aforementioned theorems imposes a degree condition on all vertices (or, on all pairs of nonadjacent vertices). In [5] and [3], a type of sufficient conditions for a digraph to be Hamiltonian was described, in which a degree condition requires only for some pairs of nonadjacent vertices. Let us recall only the following theorem of them.

Theorem 1.1 (Bang-Jensen, Gutin, H.Li [5]). *Let D be a strongly connected digraph of order $n \geq 2$. Suppose that $min\{d(x), d(y)\} \geq n - 1$ and $d(x) + d(y) \geq 2n - 1$ for any pair of non-adjacent vertices x, y with a common in-neighbour. Then D is Hamiltonian.*

A digraph D is called a bipartite digraph if there exists a partition X, Y of its vertex set into two partite sets such that every arc of D has its end-vertices in different partite sets. It is called balanced if $|X| = |Y|$.

There are analogues results to the Nash-Williams, Ghouila-Houri, Woodall, Meyniel and Thomassen theorems for balanced bipartite digraphs (see e.g., [2] and the papers cited there).

An analogous of Theorem 1.1 for bipartite digraphs was given by R. Wang [26] and recently a different result was given by Adamus [1].

Theorem 1.2 (R. Wang [26]). *Let D be a strongly connected balanced bipartite digraph of order $2a$, where $a \geq 1$. Suppose that, for every dominating pair of vertices $\{x, y\}$, either $d(x) \geq 2a - 1$ and $d(y) \geq a + 1$ or $d(y) \geq 2a - 1$ and $d(x) \geq a + 1$. Then D is Hamiltonian.*

Theorem 1.3 (Adamus [1]). *Let D be a strongly connected balanced bipartite digraph of order $2a$, where $a \geq 3$. If $d(x) + d(y) \geq 3a$ for every pair of vertices $\{x, y\}$ with a common in-neighbour or a common out-neighbour, then D is Hamiltonian.*

Let D be a balanced bipartite digraph of order $2a$, where $a \geq 2$. For integer $k \geq 0$, we say that D satisfies condition B_k when $max\{d(x), d(y)\} \geq 2a - 2 + k$ for every pair of dominating vertices x and y.

Before stating the next theorems we need to define two digraphs of order ten and eight.

Example 1. Let $D(10)$ be a bipartite digraph with partite sets

$X = \{x_0, x_1, x_2, x_3, x_4\}$ and $Y = \{y_0, y_1, y_2, y_3, y_4\}$ satisfying the following conditions: The induced subdigraph $\langle\{x_1, x_2, x_3, y_0, y_4\}\rangle$ is a complete bipartite digraph with partite sets $\{x_1, x_2, x_3\}$ and $\{y_0, y_4\}$; $\{x_1, x_2, x_3\} \to \{y_1, y_2, y_3\}$; $x_4 \leftrightarrow y_4$; $x_0 \leftrightarrow y_0$, $y_3 \to y_1$ and $x_i \leftrightarrow y_{i+1}$ for all $i \in [1, 3]$. $D(10)$ contains no other arcs. (See Figure 1, an undirected edge represents two directed arcs of opposite directions).

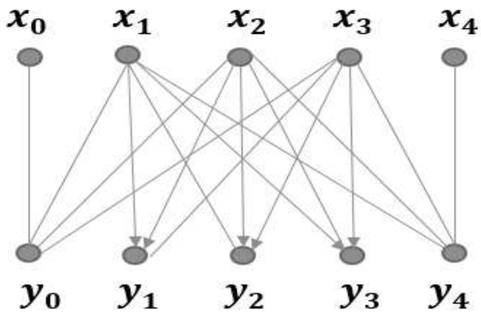

Figure 1.

It is easy to check that the digraph $D(10)$ is strongly connected and satisfies condition B_0, but the underlying undirected graph of $D(10)$ is not 2-connected and $D(10)$ has no cycle of length 8.

Example 2. Let $D(8)$ be a bipartite digraph with partite sets $X = \{x_0, x_1, x_2, x_3\}$ and $Y = \{y_0, y_1, y_2, y_3\}$, and the arc set $A(D(8))$ contains exactly the following arcs y_0x_1, y_1x_0, x_2y_3, x_3y_2 and all the arcs of the following 2-cycles: $x_i \leftrightarrow y_i$, $i \in [0, 3]$, $y_0 \leftrightarrow x_2$, $y_0 \leftrightarrow x_3$, $y_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow x_3$ (see Figure 2).

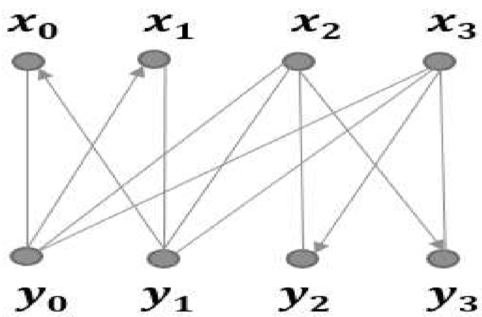

Figure 2.

Observe that $D(8)$ is not Hamiltonian.

For $a \geq 5$ Theorem 1.2 is an immediate consequence of the following theorem by the first author [14].

Theorem 1.4 (Darbinyan [14]). *Let D be a strongly connected balanced bipartite digraph of order $2a$, where $a \geq 4$. Suppose that, for every dominating pair of vertices $\{x, y\}$, either $d(x) \geq 2a-1$ or $d(y) \geq 2a-1$. Then either D is Hamiltonian or isomorphic to the digraph $D(8)$.*

A balanced bipartite digraph of order $2m$ is even pancyclic if it contains a cycle of length $2k$ for any $2 \leq k \leq m$. A cycle of a balanced bipartite digraph D is called pre-Hamiltonian if it contains all the vertices of D except two.

Characterizations of even pancyclic bipartite tournaments were given in [6] and [28]. A characterization of pancyclic ordinary k-partite ($k \geq 3$) tournaments (respectively, pancyclic ordinary complete multipartite digraphs) was established in [18] (respectively, in [19]).

Motivated by the Bondy's metaconjecture, it is natural to consider the following problem:

Problem. *Characterize those digraphs which satisfy the conditions of Theorem 1.2 (or, 1.3 or 1.4) but are not even pancyclic.*

In [15], the first author have proved the following Theorems 1.5 and 1.6.

Theorem 1.5 ([15]. *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 8$ other than a directed cycle. If $max\{d(x), d(y)\} \geq 2a - 1$ for every dominating pair of vertices $\{x, y\}$, then D contains a cycles of all even lengths less than equal $2a$ or D is isomorphic to the digraph $D(8)$.*

Theorem 1.6. ([15]. *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 8$ which contains a cycle of length $2a-2$. If $max\{d(x), d(y)\} \geq 2a-2$ for every dominating pair of vertices $\{x, y\}$, then for any k, $1 \leq k \leq a-1$, D contains a cycle of length $2k$.*

In view of Theorem 1.6 it seems quite natural to ask whether a balanced bipartite digraph of order $2a$ in which $max\{d(x), d(y)\} \geq 2a-2$ for every pair of vertices $\{x, y\}$ with a common out-neighbour contains a pre-Hamiltonian cycle (i.e., a cycle of length $2a-2$). In this paper we prove the following theorems.

Theorem 1.7. *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 10$ other than a directed cycle of length $2a$. Suppose that D satisfies condition B_0, i.e., $max\{d(x), d(y)\} \geq 2a-2$ for every dominating pair of vertices $\{x, y\}$. Then D contains a cycle of lengths $2a-2$ unless D is isomorphic to the digraph $D(10)$.*

From Theorems 1.6 and 1.7 the following theorem follows.

Theorem 1.8. *Let D be a balanced bipartite digraph of order $2a \geq 10$ other than a directed cycle of length $2a$. Suppose that D satisfies condition B_0, i.e., $max\{d(x), d(y)\} \geq 2a-2$ for every dominating pair of vertices $\{x, y\}$. Then D contains cycles of all lengths $2, 4, \ldots, 2a-2$ unless D is isomorphic to the digraph $D(10)$.*

2. TERMINOLOGY AND NOTATION

In this paper we consider finite digraphs without loops and multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and by $A(D)$, respectively. The order of D is the number of its vertices. For any $x, y \in V(D)$, we also write $x \to y$, if $xy \in A(D)$. If $xy \in A(D)$, then we say that x dominates y or y is an out-neighbour of x, and x is an in-neighbour of y. The notation $x \leftrightarrow y$ denotes that $x \to y$ and $y \to x$ ($x \leftrightarrow y$ is called a 2-cycle). We denote by $a(x, y)$ the number of arcs with end-vertices x and y. For disjoint subsets A and B of $V(D)$ we define $A(A \to B)$ as the set $\{xy \in A(D)/x \in A, y \in B\}$ and $A(A, B) = A(A \to B) \cup A(B \to A)$. If $x \in V(D)$

and $A = \{x\}$ we write x instead of $\{x\}$. If A and B are two disjoint subsets of $V(D)$ such that every vertex of A dominates every vertex of B, then we say that A dominates B, denoted by $A \rightarrow B$. $A \mapsto B$ means that $A \rightarrow B$ and there is no arc from B to A.

Let $N^+(x)$, $N^-(x)$ denote the set of out-neighbours, respectively the set of in-neighbours of a vertex x in a digraph D. If $A \subseteq V(D)$, then $N^+(x, A) = A \cap N^+(x)$ and $N^-(x, A) = A \cap N^-(x)$. The out-degree of x is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the in-degree of x. Similarly, $d^+(x, A) = |N^+(x, A)|$ and $d^-(x, A) = |N^-(x, A)|$. The degree of a vertex x in D is defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x, A) = d^+(x, A) + d^-(x, A)$).

The subdigraph of D induced by a subset A of $V(D)$ is denoted by $D\langle A\rangle$ or $\langle A\rangle$ for brevity. The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ ($m \geq 2$) and the arcs x_ix_{i+1}, $i \in [1, m-1]$ (respectively, x_ix_{i+1}, $i \in [1, m-1]$, and x_mx_1), is denoted by $x_1x_2 \cdots x_m$ (respectively, $x_1x_2 \cdots x_mx_1$). We say that $x_1x_2 \cdots x_m$ is a path from x_1 to x_m or is an (x_1, x_m)-path. If P is a path containing a subpath from x to y let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x, y]$ denotes the subpath of C from x to y. A digraph D is strongly connected (or, just, strong) if there exists an (x, y)-path in D for every ordered pair of distinct vertices x, y of D. Given a vertex x of a path P or a cycle C, we denote by x^+ (respectively, by x^-) the successor (respectively, the predecessor) of x (on P or C), and in case of ambiguity, we precise P or C as a subscript (that is x^+_P ...). Two distinct vertices x and y are adjacent if $xy \in A(D)$ or $yx \in A(D)$ (or both).

For integers a and b, $a \leq b$, let $[a, b]$ denote the set of all integers which are not less than a and are not greater than b.

Let C be a non-Hamiltonian cycle in a digraph D. An (x, y)-path P is a C-bypass if $|V(P)| \geq 3$, $x \neq y$ and $V(P) \cap V(C) = \{x, y\}$. The length of the path $C[x, y]$ is the gap of P with respect to C.

The underlying undirected graph of a digraph D is the unique graph that contains an edge xy if and only if $x \rightarrow y$ or $y \rightarrow x$ (or both).

3. PRELIMINARIES

Lemma 3.1 (Bypass Lemma 3.17, Bondy [8]). *Let D be a strongly connected digraph, and let H be a non-trivial proper subdigraph of D. If the underlying undirected graph of D is 2-connected, then D contains an H-bypass.*

Lemma 3.2 ([14]). *Let D be a strongly balanced bipartite digraph of order $2a \geq 8$ other than a directed cycle of length $2a$. If D satisfies condition B_0, then D contains a non-Hamiltonian cycle of length at least 4.*

Lemma 3.3 ([15]). *Let D be a strongly balanced bipartite digraph of order $2a \geq 8$ with partite sets X and Y. Assume that D satisfies condition B_0. Let $C = x_1y_1x_2y_2 \ldots x_ky_kx_1$ be a longest non-Hamiltonian cycle in D, where $k \geq 2$, $x_i \in X$ and $y_i \in Y$, and P be a C-bypass. If the gap of P with respect to C is equal to one, then $k = a - 1$, i.e., the longest non-Hamiltonian cycle in D has length equal to $2a - 2$.*

Theorem 3.4 ([13]). *Let D be a strongly connected balanced bipartite digraph of order $2a \geq 10$. Assume that D satisfies condition B_0. Then either the underlying undirected graph of D is 2-connected or D contains a cycle of length $2a - 2$ unless D is isomorphic to the digraph $D(10)$.*

4. THE OUTLINE OF THE PROOF OF THEOREM 1.7

Suppose, on the contrary, that a digraph D is not a directed cycle and satisfies the conditions of the theorem, but contains no cycle of length $2a - 2$. Let $C = x_1y_1x_2y_2 \ldots x_my_mx_1$ be a longest non-Hamiltonian cycle in D, where $x_i \in X$ and $y_i \in Y$ (all subscripts of the vertices x_i and y_i are taken modulo m, i.e., $x_{m+i} = x_i$ and $y_{m+i} = y_i$).

By Lemma 3.2, D contains a non-Hamiltonian cycle of length at least 4, i.e., $2 \leq m \leq a - 2$. By Theorem 3.4, the underlying undirected graph of D is 2-connected. Therefore, by Bypass Lemma, D contains a C-bypass. We choose a cycle C and a C-bypass $P := xu_1 \ldots u_sy$ such that

(i) C is a longest non-Hamiltonian cycle in D;

(ii) the gap of C-bypass P with respect to C is minimum subject to (i);

(iii) the length of P is minimum subject to (i) and (ii).

Without loss of generality, we assume that $x := x_1$. Let $R := V(D) \setminus V(C)$, $P_1 := P[u_1, u_s]$ and $C' := V(C[y_1, y^-_C])$. Note that $|R| \geq 4$. From Lemma 3.3 it follows that $|V(C[x, y])| \geq 3$. Then $|C'| \geq s$ since C is a longest non-Hamiltonian cycle in D.

Firstly we prove that $|V(P_1)| = 1$, i.e., $s = 1$. From $s = 1$ it follows that $u_1 \in Y$ and $y \in X$ since $x_1 \in X$. Without loss of generality we may assume that $y := x_r$. From now on, let $y := u_1$, and we divide the proof of the theorem into two parts: $|C'| = 1$ and $|C'| \geq 2$.

Part I. $|C'| = 1$, i.e., $r = 2$ and $x_1 \rightarrow y \rightarrow x_2$.

By condition B_0, $max\{d(y), d(y_1)\} \geq 2a - 2$ since $\{y, y_1\} \rightarrow x_2$. Without loss of generality, assume that $d(y) \geq 2a - 2$. For this part we first prove Claims 1-5 below.

Claim 1. If $x \in R \cap X$ and $x \leftrightarrow y$, then $d(x) \leq 2a - 3$ and $d(x_1) \geq 2a - 2$.

Claim 2. There are no two distinct vertices $x, x_0 \in R \cap X$ such that $x \leftrightarrow y$ and $x_0 \leftrightarrow y$.

Claim 3. If $y \leftrightarrow x$ for some $x \in R \cap X$, then $a(y, z) = 1$ for all $z \in R \cap X \setminus \{x\}$.

Claim 4. If $y \leftrightarrow x$ for some $x \in R \cap X$, then $d^-(y, R \cap X \setminus \{x\}) = 0$.

Claim 5. There is no $x \in X \cap R$ such that $y \leftrightarrow x$, i.e., in subdigraph $D\langle R\rangle$ through the vertex y there is no cycle of length two.

Now we can finish the discussion of Part I. From Claim 5 it follows that in $D\langle R\rangle$ there is no cycle of length two through the vertex y. Then, since $d(y) \geq 2a-2$ and $|R| \geq 4$, it follows that $|R| = 4$. Put $X \cap R = \{x, x_0\}$ and $Y \cap R = \{y, y_0\}$. Then $a(y,x) = a(y,x_0) = 1$ and the vertex y and every vertex of $X \cap C$ form a 2-cycle, i.e.,

$$y \leftrightarrow \{x_1, x_2, \ldots, x_m\}. \tag{1}$$

First consider the case $d^+(y, \{x, x_0\}) \geq 1$. Assume, without loss of generality, that $y \mapsto x$. Then, by (1), $d^+(x, \{y_1, y_2, \ldots, y_m\}) = 0$. Together with $xy \notin A(D)$ this implies that $x \to y_0$ since D is strong. Therefore, $d(x) \leq 2a-3$ since $|Y \cap C| \geq 2$. By (1), it is clear that $d^+(y_0, \{x_1, x_2, \ldots, x_m\}) = 0$. If $y \to x_0$, then analogously we obtain that $d^+(x_0, \{y, y_1, y_2, \ldots, y_m\}) = 0$ and $x_0 \to y_0$, $d(x_0) \leq 2a-3$, which contradicts condition B_0 since $max\{d(x), d(x_0)\} \leq 2a-3$ and $\{x, x_0\} \to y_0$. We may assume therefore that $yx_0 \notin A(D)$. Then $x_0 \to y$ (by $a(y,x_0) = 1$), $d^-(x_0, \{y, y_1, y_2, \ldots, y_m\}) = 0$ (by (1)) and hence, $y_0 \to x_0$ since D is strong. Now it is not difficult to show that

$$d^-(y_0, \{x_1, x_2, \ldots, x_m\}) = d^-(x_0, \{y_1, y_2, \ldots, y_m\})$$

$$= d^+(x, \{y_1, y_2, \ldots, y_m\}) = 0.$$

Therefore, $d(x_0) \leq 2a-3$ and $d(y_i) \leq 2a-3$. Since for all $i \in [1,m]$, $\{x_i, x_0\} \to y$ and $d(x_0) \leq 2a-3$, from condition B_0 it follows that $d(x_i) \geq 2a-2$ for all $i \in [1,m]$. This together with $a(x_i, y_0) = 0$ (by (16) and the last equalities) imply that $y_i \leftrightarrow x_i$. Thus, $\{y_{i-1}, y_i\} \to x_i$ and $max\{d(y_{i-1}), d(y_i)\} \leq 2a-3$, which is a contradiction.

Now consider the case $d^+(y, \{x, x_0\}) = 0$. Then $\{x, x_0\} \to y$ because of $a(y,x) = a(y,x_0) = 1$, i.e., $\{x, x_0\}$ is a dominating pair. It is clear that $d^-(x, \{y_1, y_2\}) = d^-(x_0, \{y_1, y_2\}) = 0$. This together with $d^+(y, \{x_0, x\}) = 0$ imply that $max\{d(x), d(x_0)\} \leq 2a-3$, which is a contradiction because of $\{x, x_0\} \to y$. This completes the discussion of the part $|C'| = 1$.

Part 2. $|C'| \geq 2$.

Then $|C'| \geq 3$ since $|C'|$ is odd. For this part we first will prove Claims 6-7.

Claim 6. If $|C'| \geq 3$, then

(i). $d(y) \leq 2a-3$ and $d(y_{r-1}) \geq 2a-2$;

(ii). there is no $x \in X \cap R$ such that $x \leftrightarrow y_{r-1}$; i.e., $a(y_{r-1}, x) \leq 1$ for all $x \in X \cap R$;

(iii). $a(y_{r-1}, x) = 1$ for all $x \in X \cap R$, $|R| = 4$, $d(y_{r-1}) = 2a-2$ and the vertex y_{r-1} together with every vertex of $X \cap V(C)$ forms a 2-cycle. In particular, $x_r \leftrightarrow y_{r-1}$ and $y_{r-1} \leftrightarrow x_2$;

(iv). $d^-(y_{r-1}, \{x, x_0\}) = 0$ and $y_{r-1} \mapsto \{x, x_0\}$, where $X \cap R = \{x, x_0\}$ and $Y \cap R = \{y, y_0\}$;

(v). $max\{d(x_0), d(x)\} \leq 2a-3$ and $d^-(v, \{x_0, x\}) \leq 1$ for all $v \in Y$;

(vi). $|C'| = 3$, i.e., $r = 3$.

From Claims 6(iii), 6(iv) and 6(v) it follows that

(a) *If* $|C'| \geq 3$, *then* $|C'| = 3$, $|R| = 4$, $y_2 \mapsto \{x, x_0\}$, $d(y) \leq 2a-3$, $d(y_2) = 2a-2$ *and the vertex* y_2 *together with every vertex of* $X \cap V(C)$ *forms a 2-cycle.*

Claim 7. If $|C'| = 3$, then

(i). $d(y_0) \leq 2a-3$, (recall that $\{y_0\} = Y \cap R \setminus \{y\}$);

(ii). $d(x_2) \leq 2a-3$ and $d(x_3) \geq 2a-2$;

(iii). $d(y_1) \geq 2a-2$ and

(iv). $x \to y_k$ or $x_0 \to y_k$, where $k \in [3,m]$, then $x_k y_1 \notin A(D)$.

From Claims 6(i), 6(iv) and 7(i) it follows that if $|C'| = 3$, then

$$max\{d(y), d(y_0), d(x), d(x_0)\} \leq 2a-3, \tag{2}$$

in particular, by condition B_0, we have

$$max\{d^-(u, \{y, y_0\}), d^-(v, \{x, x_0\})\} \leq 1$$

$$\text{for all} \quad u \in X \quad \text{and} \quad v \in Y. \tag{3}$$

Now we are ready to complete the proof of Theorem. Combining (2) and Claim 7(ii), we obtain

$$max\{d(x), d(x_0), d(y), d(y_0), d(x_2)\} \leq 2a-3. \tag{4}$$

This and condition B_0 imply that $d^-(v, \{x, x_0, x_2\}) \leq 1$ for all $v \in Y$. Therefore, since $d(y_1) \geq 2a-2$ (Claim 7(iii)), the vertex y_1 and every vertex of $X \setminus \{x, x_0, x_2\}$ form a 2-cycle, i.e.,

$$y_1 \leftrightarrow x_i \quad \text{for all} \quad i \in [1,m] \setminus \{2\}. \tag{5}$$

Now using Claim 7(iv), we obtain $A(\{x, x_0\} \to \{y_2, y_3, \ldots, y_m\}) = \emptyset$. From $y_2 \mapsto \{x, x_0\}$ (Claim 6(iv)) and the minimality of the gap $|C'|+1$ it follows that $d^-(y, \{x, x_0\}) = 0$. The last two equalities imply that

$$A(\{x, x_0\} \to \{y, y_2, y_3, \ldots, y_m\}) = \emptyset. \tag{6}$$

By (3), in particular, we have

$$max\{d^-(y_1, \{x, x_0\}), d^-(y_0, \{x, x_0\})\} \leq 1. \tag{7}$$

Since D is strong, from (6) and (7) it follows that $x \to y_0$ or $x_0 \to y_0$. Again using (7), we obtain that if $x \to y_0$, then $x_0 y_0 \notin A(D)$ and $x_0 \to y_1$; if $x_0 \to y_0$, then $xy_0 \notin A(D)$ and $x \to y_1$.

Because of the symmetry between x and x_0, we can assume that $x_0 \to y_0$, $x \to y_1$ and $x_0 y_1 \notin A(D)$, $xy_0 \notin A(D)$. It is not difficult to show that x_2 and every vertex y_i with $i \in [3,m]$ are not adjacent. Indeed, if $y_i \to x_2$, then, by (5), $y_1 \to x_{i+1}$ and hence, $x_1 y x_3 y_3 \ldots y_i x_2 y_2 x y_1 x_{i+1} \ldots x_1$ is a cycle of length $2a-2$; if $x_2 \to y_i$, then, by (a), $x_i \to y_2$ and hence, $x_1 y x_3 y_3 \ldots x_i y_2 x y_1 x_2 y_i \ldots y_m x_1$ is a cycle of length $2a-2$. Thus, in both cases we have a contradiction.

Therefore, $a(y_i, x_2) = 0$ for all $i \in [3,m]$. This and (3) imply that $d(y_i) \leq 2a-3$ for all $i \in [3,m]$. From $y \to x_3$, $d(y) \leq 2a-3$ (Claim 6(i)) and condition B_0 it follows that $d^-(x_3, \{y_0, y_3, y_4, \ldots y_m\}) = 0$. Now from $d(x_3) \geq 2a-2$ (Claim 7(ii)), we have that $m = 3$, i.e., $a = 5$. Since $x \to y_1$, $x_0 \to y_0$ and (4), it follows that $d^+(x_2, \{y_0, y_1\}) = 0$. Because of $d(y_1) \geq 2a - 2$ (Claim 7(iii)) and $d^-(y_1, \{x_0, x_2\}) = 0$ we have $y_1 \to x_0$, $x_3 \leftrightarrow y_1$ and $y_1 \to x_1$. From this it follows that $y_0x_2 \notin A(D)$ (for otherwise, $y_1x_0y_0x_2y_2 \ldots x_1y_1$ is a cycle of length $2a - 2$, a contradiction). Therefore, $a(x_2, y_0) = 0$. If $y_0 \to x_1$, then the cycle $x_1yx_3y_1x_2y_2x_0y_0x_1$ is a cycle of length $2a - 2 = 8$, a contradiction. Therefore, $y_0x_1 \notin A(D)$. So, we have $d^+(y_0, \{x_1, x_2, x_3\}) = 0$. Then $d^+(y_0, \{x, x_0\}) \geq 1$ since D is strong. It is easy to see that $y_0x \notin A(D)$ (for otherwise, $x_1yx_3y_2x_0y_0xy_1x_1$ is a cycle of length 8, a contradiction. Therefore, $d^+(y_0, \{x_1, x_2, x_3, x\}) = 0$. On the other hand, from (6) and $x_0y_1 \notin A(D)$ we have $N^+(x_0) = \{y_0\}$. Now it is not difficult to see that there is no path from x_0 to any vertex of $V(C)$ since $N^+(y_0) = \{x_0\}$, which contradicts that D is strong. So, the discussion of the case $|C'| \geq 3$ is completed. Theorem 1.7 is proved.

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