A Theorem on Higher-order Differences of Two-state Markov Chains

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ABSTRACT
We present a theorem on existence of limits of higher-order absolute differences, taken from progressive terms of a given homogenous two-state Markov chain, when the order of differences converges to infinity.

Keywords
Higher-order differences, two-state Markov chain

1. INTRODUCTION
The difference analysis suggested in [1]-[6] is a method for studying irregular or random time series, based on consideration of higher absolute differences taken from the series' progressive terms.

This method allowed us to reveal some new aspects in dynamical systems and random processes. Thus, a difference analog for well-known in deterministic chaos Lyapunov exponent in [3] is suggested, in [6] the bistability of higher-order differences taken from discrete-time signals is proved and some applications to digital communication and signal processing are outlined.

In this report an application of difference analysis to finite Markov chains is considered: a limiting theorem for higher-order differences of homogenous two-state Markov chains is presented.

2. FORMULATION OF MAIN THEOREM
2.1 Some notions
We consider discrete-time random processes

$$\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_n, \ldots)$$

whose components ξ_n take binary values 0 and 1 with some positive probabilities p_n and q_n, $p_n + q_n = 1$. Then for every $k \geq 1$ the differences $\xi_n^{(k)} = |\xi_{n+1}^{(k-1)} - \xi_n^{(k-1)}|$, $(n \geq 0, k \geq 1, \xi_n^{(0)} = \xi_n)$ also take binary values with some positive probabilities $p_n^{(k)}$ and $q_n^{(k)}$, and hence, one can consider the random difference processes

$$\boldsymbol{\xi}^{(k)} = (\xi_0^{(k)}, \xi_1^{(k)}, \ldots, \xi_n^{(k)}, \ldots).$$

We are interested in existence of the limit of kth order difference processes $\boldsymbol{\xi}^{(k)}$ when k goes to infinity. Let an infinite subset of natural series $\Lambda \subseteq \mathbb{N}$ be given. We say that $\boldsymbol{\xi}^{(k)}$ converge to the random binary sequence $\boldsymbol{\xi}^{(\infty)}$, if $p_n^{(k)}$ tend to some numbers $p_n^{(\infty)}$ as $k \to \infty$ and $k \in \Lambda$ (convergence by probability on Λ); clearly, the final probabilities may depend on Λ, $p_n^{(\infty)} = p_n^{(\infty)}(\Lambda)$. Then the final process

$$\boldsymbol{\xi}^{(\infty)} = \boldsymbol{\xi}_\Lambda^{(\infty)} = (\xi_0^{(\infty)}, \xi_1^{(\infty)}, \ldots, \xi_n^{(\infty)}), \quad \xi_n^{(\infty)} = \xi_n^{(\infty)}(\Lambda)$$

is also a discrete random process, whose components take the values 0 and 1 with some "final" probabilities $p_n^{(\infty)}$ and $q_n^{(\infty)}$. We note that when infinite $\Lambda \subseteq \mathbb{N}$ is chosen arbitrarily, the final process $\boldsymbol{\xi}_\Lambda^{(\infty)}$ may not exist [5].

In [4, 5] the application of difference analysis to studying independent random binary sequences has been considered. Theorem 1, formulated in the next Section 2.2, relates to discrete-time binary Markov processes.

2.2 Two-state Markov chains
In this report we consider two-state Markov processes $\boldsymbol{\xi} = (\xi_n)_{n=0}^{\infty}$: the state space $X = \{x\}$ consists of two items, and we suppose that x takes binary values 0 and 1, that is $X = \{0, 1\}$ and components ξ_n of $\boldsymbol{\xi}$ are binary random variables. We denote one-step transition probabilities as

$$\pi_n(x, y) = P(\xi_n = y | \xi_{n-1} = x) \quad (x, y \in X, n = 1, 2, \ldots).$$

Let the initial distribution $P(\xi_0 = x)$ on X be given and

$$p_n(x) = P(\xi_n = x)$$

denotes the probability of assuming of the given state $x \in X$ at nth step ξ_n of the process $\boldsymbol{\xi} = (\xi_n)_{n=0}^{\infty}$. We use the notation $p_n = P(\xi_n = 0)$, $q_n = P(\xi_n = 1)$; our Theorem 1 concerns the processes $\boldsymbol{\xi}$ and their kth order differences $\boldsymbol{\xi}^{(k)}$, as defined in Section 2.1.

If infinite Λ and binary Markov chain $\boldsymbol{\xi}$ are arbitrary, the studying of limiting transition of $p_n^{(k)}$, when k goes to ∞ along Λ, can be complicated. Moreover, even if $\boldsymbol{\xi}$ is homogenous (one-step transition probabilities π_n do not depend on n) but $\Lambda \subseteq \mathbb{N}$ is chosen arbitrarily, the final process $\boldsymbol{\xi}_\Lambda^{(\infty)}$ may not exist. The next Theorem 1 claims the existence of the final processes $\boldsymbol{\xi}_\Lambda^{(\infty)}$ for homogenous binary Markov chains $\boldsymbol{\xi}$ and some infinite $\Lambda \subseteq \mathbb{N}$: the theorem asserts the convergence (by probability) of kth-order absolute difference processes $\boldsymbol{\xi}^{(k)}$ as $k \to \infty$ and $k \in \Lambda_0$, where

$$\Lambda_0 = \{2^m - 1 : m = 1, 2, \ldots\}.$$

The theorem is valid under some natural restrictions on one-step transition probabilities (we omit the details).

Theorem 1. *Let the random sequence $\boldsymbol{\xi} = (\xi_n)_{n=0}^{\infty}$ be the homogenous binary Markov chain. Then under some restrictions on one-step transition probabilities the final process $\boldsymbol{\xi}_{\Lambda_0}^{(\infty)}$ exists and it is identically distributed binary* *sequence, i.e., the following limits*

$$p_n^{(\infty)} = p_n^{(\infty)}(\Lambda_0) = \lim_{\substack{k\to\infty \\ k\in\Lambda_0}} p_n^{(k)},\ q_n^{(\infty)} = q_n^{(\infty)}(\Lambda_0) = \lim_{\substack{k\to\infty \\ k\in\Lambda_0}} q_n^{(k)}$$

exist. Here, $p_n^{(\infty)} = p_n^{(\infty)}(x)$, $q_n^{(\infty)} = q_n^{(\infty)}(x)$, $p_n^{(k)} = p_n^{(k)}(x)$, $q_n^{(k)} = q_n^{(k)}(x)$ $(x \in X)$, $p_n^{(\infty)} + q_n^{(\infty)} = 1$, *and final probabilities* $p_n^{(\infty)}$ *and* $q_n^{(\infty)}$ *do not depend on* n.

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