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A Structure of the Subgraph Induced at a Labeling of a Graph by the Subset of Vertices with an Interval Spectrum

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Abstract

The sets of vertices and edges of an undirected, simple, finite, connected graph G are denoted by $V(G)$ and $E(G)$, respectively. An arbitrary nonempty finite subset of consecutive integers is called an interval. An injective mapping $\varphi : E(G) \to \{1, 2, \dots, |E(G)|\}$ is called a labeling of the graph G. If G is a graph, x is its arbitrary vertex, and φ is its arbitrary labeling, then the set $S_G(x, \varphi) \equiv \{\varphi(e)/e \in E(G), e$ is incident with $x\}$ is called a spectrum of the vertex x of the graph G at its labeling φ. For any graph G and its arbitrary labeling φ, a structure of the subgraph of G, induced by the subset of vertices of G with an interval spectrum, is described.

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1 Introduction

We consider undirected, simple, finite and connected graphs, containing at least one edge. The terms and concepts which are not defined can be found in [2].

For a graph G, we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. The set of vertices of G adjacent to a vertex $x \in V(G)$ is denoted by $I_G(x)$. The set of edges of G incident with a vertex $x \in V(G)$ is denoted by $J_G(x)$.

If G is a graph, and $x \in V(G)$, then $d_G(x)$ denotes the degree of the vertex x in the graph G. For any graph G, $\Delta(G)$ denotes the maximum degree of a vertex of G.

For any graph G, we define the subsets $V'(G)$ and $V''(G)$ of its vertices by the following way: $V'(G) \equiv \{x \in V(G)/\ d_G(x) = 1\}$, $V''(G) \equiv \{x \in V(G)/\ d_G(x) \geq 2\}$.

The distance in a graph G between its vertices x and y is denoted by $d_G(x, y)$.

For any graph G, we denote by $diam(G)$ its diameter. A vertex $x \in V(G)$ is called a peripheral vertex of a graph G if there exists a vertex $y \in V(G)$ satisfying the condition $d_G(x, y) = diam(G)$.

For an arbitrary nonempty finite subset A of the set $\mathbb{Z}_+$, we denote by $l(A)$ and $L(A)$, respectively, the least and the greatest element of A.

An arbitrary nonempty finite subset A of the set $\mathbb{Z}_+$ is called an interval if it satisfies the condition $|A| = L(A) - l(A) + 1$. An interval with the minimum element p and the maximum element q is denoted by $[p, q]$.

An injective mapping $\varphi : E(G) \to \mathbb{N}$ is called a labeling of the graph G. For any graph G, we denote by $\psi(G)$ the set of all labelings of the graph G.

If G is a graph, $\varphi \in \psi(G)$, and $E_0 \subseteq E(G)$, we set

$$\varphi[E_0] \equiv \bigcup_{e \in E_0} \{\varphi(e)\}.$$

If G is a graph, $x \in V(G)$, and $\varphi \in \psi(G)$, then the set $\varphi[J_G(x)]$ is called a spectrum of the vertex x of the graph G at the labeling φ.

If G is a graph, and $\varphi \in \psi(G)$, then we set $U(G, \varphi) \equiv \{x \in V(G)/\ \varphi[J_G(x)]$ is an interval$\}$.

If G is a graph, and $\varphi \in \psi(G)$, then we denote by $G^{(\varphi,int)}$ the subgraph of the graph G induced by the subset $U(G, \varphi)$ of its vertices.

For any graph G, we set $\lambda(G) \equiv \{\varphi \in \psi(G)/\ U(G, \varphi) \neq \emptyset\}$. Clearly, for any graph G, $\lambda(G) \neq \emptyset$.

If G is a graph, $\varphi \in \lambda(G)$, $(x_0, x_1) \in E(G)$, then the simple path $P = (x_0, (x_0, x_1), x_1)$ is called a trivial φ-gradient path of G iff the following two conditions hold:

1. $\{x_0, x_1\} \subseteq U(G, \varphi)$,
2. at least one of the following two conditions holds:
   (a) $\varphi((x_0, x_1)) = L(\varphi[J_{G(\varphi,int)}(x_0)]) = l(\varphi[J_{G(\varphi,int)}(x_1)])$,
   (b) $\varphi((x_0, x_1)) = l(\varphi[J_{G(\varphi,int)}(x_0)]) = L(\varphi[J_{G(\varphi,int)}(x_1)])$.

If G is a graph, $\varphi \in \lambda(G)$, then a simple path $P = (x_0, (x_0, x_1), x_1, \ldots, x_k, (x_k, x_{k+1}), x_{k+1})$ with $k \in \mathbb{Z}_+$ is called a φ-gradient path of G, if either $k = 0$ and P is a trivial φ-gradient path of G, or $k \in \mathbb{N}$ and the following two conditions hold:

1. $V(P) \subseteq U(G, \varphi)$,
2. exactly one of the following two conditions holds:
   (a) for any $i \in [0, k]$, $\varphi((x_i, x_{i+1})) = L(\varphi[J_{G(\varphi,int)}(x_i)]) = l(\varphi[J_{G(\varphi,int)}(x_{i+1})])$,
   (b) for any $i \in [0, k]$, $\varphi((x_i, x_{i+1})) = l(\varphi[J_{G(\varphi,int)}(x_i)]) = L(\varphi[J_{G(\varphi,int)}(x_{i+1})])$.

If G is a graph, $\varphi \in \lambda(G)$, then the set of all φ-gradient paths of G is denoted by $\xi(G, \varphi)$.

If G is a graph, $\varphi \in \lambda(G)$, and $P \in \xi(G, \varphi)$, then P is called a maximal φ-gradient path of G, if there is no $\widetilde{P} \in \xi(G, \varphi)$ with $V(P) \subset V(\widetilde{P})$.

If G is a graph, $\varphi \in \lambda(G)$, then the set of all maximal φ-gradient paths of G is denoted by $\tau(G, \varphi)$.

For arbitrary integers n and i, satisfying the inequalities $n \geq 3$, $2 \leq i \leq n-1$, and for any sequence $A_{n-2} \equiv (a_1, a_2, \ldots, a_{n-2})$ of nonnegative integers, we define the sets $V[i, A_{n-2}]$ and $E[i, A_{n-2}]$ as follows:

$$V[i, A_{n-2}] \equiv \begin{cases} \{y_{i,1}, \ldots, y_{i,a_{i-1}}\}, & \text{if } a_{i-1} > 0 \\ \emptyset, & \text{if } a_{i-1} = 0, \end{cases}$$
$$E[i, A_{n-2}] \equiv \begin{cases} \{(x_i, y_{i,j}), /\ 1 \leq j \leq a_{i-1}\}, & \text{if } a_{i-1} > 0 \\ \emptyset, & \text{if } a_{i-1} = 0. \end{cases}$$

For any integer $n \geq 3$, and for any sequence $A_{n-2} \equiv (a_1, a_2, \ldots, a_{n-2})$ of nonnegative integers, we define a graph $T[A_{n-2}]$ as follows:

$$V(T[A_{n-2}]) \equiv \{x_1, \ldots, x_n\} \cup \left(\bigcup_{i=2}^{n-1} V[i, A_{n-2}]\right),$$
$$E(T[A_{n-2}]) \equiv \{(x_i, x_{i+1})/\ 1 \leq i \leq n-1\} \cup \left(\bigcup_{i=2}^{n-1} E[i, A_{n-2}]\right).$$

A graph G is called a galaxy, if either $G \cong K_2$, or there exist an integer $n \geq 3$ and a sequence $A_{n-2} \equiv (a_1, a_2, \ldots, a_{n-2})$ of nonnegative integers, for which $G \cong T[A_{n-2}]$.

In the paper, for any graph G and arbitrary $\varphi \in \lambda(G)$, a structure of the subgraph $G^{(\varphi,int)}$ of the graph G is described. The main result was announced in [1].

2 Preliminary Notes

Lemma 2.1 *If G is a graph, $\varphi \in \lambda(G)$, $\{x, y\} \subseteq U(G, \varphi)$, $(x, y) \in E(G)$, then $|\varphi[J_G(x)] \cap \varphi[J_G(y)]| = 1$.*

Proof. If $\min\{d_G(x), d_G(y)\} = 1$, the statement is evident. Now suppose that $\min\{d_G(x), d_G(y)\} \geq 2$. Since $(x, y) \in E(G)$ we have $|\varphi[J_G(x)] \cap \varphi[J_G(y)]| \geq 1$. Let us assume that $|\varphi[J_G(x)] \cap \varphi[J_G(y)]| \geq 2$. It means that there exist $e' \in J_G(x)$, $e'' \in J_G(y)$, which satisfy the conditions $e' \neq (x, y)$, $e'' \neq (x, y)$, $e' \neq e''$, $\varphi(e') = \varphi(e'')$. It is incompatible with $\varphi \in \lambda(G)$.

Lemma is proved.

Corollary 2.2 *Let G be a graph, and $\varphi \in \lambda(G)$. Suppose that vertices $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ of the graph G satisfy the conditions*

1. *$\{x_1, \ldots, x_n\} \subseteq U(G, \varphi) \cap V''(G)$,*
2. *for any $i \in [1, n-1]$, $(x_i, x_{i+1}) \in E(G)$.*

Then exactly one of the following two statements is true:

1. *for any $i \in [1, n-1]$, $\varphi((x_i, x_{i+1})) = L(\varphi[J_G(x_i)]) = l(\varphi[J_G(x_{i+1})])$,*
2. *for any $i \in [1, n-1]$, $\varphi((x_i, x_{i+1})) = l(\varphi[J_G(x_i)]) = L(\varphi[J_G(x_{i+1})])$.*

Corollary 2.3 *If G is a graph, and $\varphi \in \lambda(G)$, then $G^{(\varphi,int)}$ is a forest.*

Proof. Assume the contrary: the graph $G^{(\varphi,int)}$ contains a subgraph G_0 which is isomorphic to a simple cycle. Clearly, there exists an edge $e_0 = (x, y) \in E(G_0)$, for which $\varphi(e_0) = l(\varphi[E(G_0)])$. From here, taking into account that $\{x, y\} \subseteq U(G, \varphi) \cap V''(G)$, we obtain $|\varphi[J_G(x)] \cap \varphi[J_G(y)]| \geq 2$. It contradicts lemma 2.1.

Corollary is proved.

Lemma 2.4 *Let G be a graph, and $\varphi \in \lambda(G)$. Then, for an arbitrary vertex $x \in U(G, \varphi)$, the inequality $|I_G(x) \cap U(G, \varphi) \cap V''(G)| \leq 2$ holds.*

Proof. Suppose that there exists a vertex $z_0 \in U(G,\varphi)$ with $|I_G(z_0) \cap U(G,\varphi) \cap V''(G)| \geq 3$. Let us choose three different vertices y_1, y_2, y_3 from the set $I_G(z_0) \cap U(G,\varphi) \cap V''(G)$.

Note that the vertices y_1, z_0, y_2 of the graph G satisfy the conditions of corollary 2.2 (with y_1 in the role of x_1, z_0 in the role of x_2, y_2 in the role of x_3, and with $n = 3$).

Note also that the vertices y_1, z_0, y_3 of the graph G satisfy the conditions of corollary 2.2 (with y_1 in the role of x_1, z_0 in the role of x_2, y_3 in the role of x_3, and with $n = 3$).

Case 1. For the vertices y_1, z_0, y_2 of the graph G, the statement 1) of corollary 2.2 is true. It means that $\varphi((y_1,z_0)) = L(\varphi[J_G(y_1)]) = l(\varphi[J_G(z_0)])$, $\varphi((z_0,y_2)) = L(\varphi[J_G(z_0)]) = l(\varphi[J_G(y_2)])$. It is not difficult to see that for the vertices y_1, z_0, y_3 of the graph G also the statement 1) of corollary 2.2 is true. It means that $\varphi((y_1,z_0)) = L(\varphi[J_G(y_1)]) = l(\varphi[J_G(z_0)])$, $\varphi((z_0,y_3)) = L(\varphi[J_G(z_0)]) = l(\varphi[J_G(y_3)])$. The equalities $\varphi((z_0,y_2)) = L(\varphi[J_G(z_0)])$ and $\varphi((z_0,y_3)) = L(\varphi[J_G(z_0)])$ are incompatible.

Case 2. For the vertices y_1, z_0, y_2 of the graph G, the statement 2) of corollary 2.2 is true.

The proof is similar as in case 1.

Lemma is proved.

Lemma 2.5 *Let G be a graph, $\varphi \in \lambda(G)$, $\xi(G,\varphi) \neq \emptyset$, $P \in \xi(G,\varphi)$. Then there exists a unique $\widetilde{P} \in \tau(G,\varphi)$ satisfying the condition $V(P) \subseteq V(\widetilde{P})$.*

Proof is evident.

Corollary 2.6 *Let G be a graph, $\varphi \in \lambda(G)$, $\{x,y\} \subseteq V''(G^{(\varphi,int)})$, $(x,y) \in E(G)$. Then there exists a unique $\widetilde{P} \in \tau(G,\varphi)$ satisfying the condition $\{x,y\} \subseteq V(\widetilde{P})$.*

3 Main Result

Theorem 3.1 *[1] For any graph G and arbitrary $\varphi \in \lambda(G)$, $G^{(\varphi,int)}$ is a forest, each connected component H of which satisfies one of the following two conditions: 1) $H \cong K_1$, and the only vertex of the graph H may or may not belong to the set $V'(G)$, 2) H is a galaxy satisfying one of the following three conditions: a) $V'(H) \subseteq V'(G)$, b) exactly one vertex of the set $V'(H)$, which is a peripheral vertex of H, doesn't belong to the set $V'(G)$, c) exactly two vertices of the set $V'(H)$, with diam(H) as the distance between them, don't belong to the set $V'(G)$.*

Proof. Choose an arbitrary $\varphi \in \lambda(G)$. Let us consider an arbitrary connected component H of the graph $G^{(\varphi,int)}$. By corollary 2.3, H is a tree.

Case 1. $|V(H)| = 1$. In this case there is nothing to prove.

Case 2. $|V(H)| = 2$. Clearly, $H \cong K_2$, and the proposition is evident.

Case 3. $|V(H)| \geq 3$. Clearly, $|V''(H)| \geq 1$.

Case 3.1. $|V''(H)| = 1$. In this case $H \cong K_{\Delta(H),1}$, $|V'(H)| = \Delta(H) = |V(H)| - 1 \geq 2$, $diam(H) = 2$. Clearly, H is a galaxy, and all vertices of $V'(H)$ are peripheral vertices of the graph H. Without loss of generality we can assume that $V''(H) = \{u_0\}$, and, moreover, that the vertices $u' \in V'(H)$ and $u'' \in V'(H)$ satisfy the conditions $l(\varphi[E(H)]) = \varphi((u_0, u'))$, $L(\varphi[E(H)]) = \varphi((u_0, u''))$.

Now consider an arbitrary vertex $z \in V'(H)$, which satisfies the condition $l(\varphi[E(H)]) < \varphi((u_0, z)) < L(\varphi[E(H)])$.

Let us show that $z \in V'(G)$. Assume the contrary: $z \in V''(G)$. From here we obtain the inequality $|\varphi[J_G(u_0)] \cap \varphi[J_G(z)]| \geq 2$, which contradicts lemma 2.1.

Consequently, $V'(H) \cap V''(G) \subseteq \{u', u''\}$, and, therefore, $0 \leq |V'(H) \cap V''(G)| \leq 2$. It completes the proof of case 3.1.

Case 3.2. $|V''(H)| \geq 2$. Clearly, in this case there exist vertices $x \in V''(H)$ and $y \in V''(H)$ satisfying the condition $(x, y) \in E(H)$. By corollary 2.6, there exists a unique $P_0 \in \tau(G, \varphi)$ with $\{x, y\} \subseteq V(P_0)$. Suppose that w' and w'' are endpoints of P_0. It is not difficult to see that $d_{P_0}(w', w'') \geq 3$ and $|I_G(w') \cap U(G, \varphi)| = |I_G(w'') \cap U(G, \varphi)| = 1$.

Let us show that

$$\Big(\bigcup_{x \in V''(P_0)} I_H(x)\Big) \backslash V(P_0) \subseteq V'(G).$$

If $(\bigcup_{x \in V''(P_0)} I_H(x)) \backslash V(P_0) = \emptyset$, the required relation is evident. Now assume that $(\bigcup_{x \in V''(P_0)} I_H(x)) \backslash V(P_0) \neq \emptyset$.

Choose an arbitrary vertex $z \in (\bigcup_{x \in V''(P_0)} I_H(x)) \backslash V(P_0)$. Let us show that $z \in V'(G)$. Assume the contrary: $z \in V''(G)$.

Consider the vertex $z_0 \in V''(P_0)$ which is adjacent to z. From the properties of P_0 it follows that $l(\varphi[J_H(z_0)]) < \varphi((z_0, z)) < L(\varphi[J_H(z_0)])$. Since $\{z_0, z\} \subseteq U(G, \varphi)$ we obtain that $|\varphi[J_G(z_0)] \cap \varphi[J_G(z)]| \geq 2$. It contradicts lemma 2.1.

Thus, indeed, $(\bigcup_{x \in V''(P_0)} I_H(x)) \backslash V(P_0) \subseteq V'(G)$. It implies that $V'(H) \cap V''(G) \subseteq \{w', w''\}$, $0 \leq |V'(H) \cap V''(G)| \leq 2$, and, that P_0 is the unique path in the graph H between its vertices w' and w''. Now it is easy to see that $diam(H) = diam(P_0) = d_{P_0}(w', w'') = d_H(w', w'')$. It completes the proof of case 3.2.

Theorem is proved.

Corollary 3.2 *[1] If G is a graph with $V'(G) = \emptyset$, and $\varphi \in \lambda(G)$, then an arbitrary connected component of the forest $G^{(\varphi,int)}$ is a simple path.*

Corollary 3.3 *[1] A labeling, which provides every vertex of a graph G with an interval spectrum, exists iff G is a galaxy.*

Corollary 3.4 *If $n \in \mathbb{N}$, and $\varphi \in \lambda(K_n)$, then the forest $K_n^{(\varphi,int)}$ is a tree which is isomorphic either to K_1, or to K_2.*

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