Applied Mathematical Sciences, Vol. 9, 2015, no. 5, 243 - 253
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2015.411972

On the Set of Simple Hypergraph Degree Sequences

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Abstract

For a given m, $0 < m \leq 2^n$, let $D_m(n)$ denote the set of all hypergraphic sequences for hypergraphs with n vertices and m hyperedges. A hypergraphic sequence in $D_m(n)$ is upper hypergraphic if all its components are at least $m/2$. Let $\widehat{D}_m(n)$ denote the set of all upper hypergraphic sequences. A structural characterization of the lowest and highest rank maximal elements of $\widehat{D}_m(n)$ was provided in an earlier study. In the current paper we present an analogous characterization for all upper non-hypergraphic sequences. This allows determining the thresholds $\bar{r}_{min}$ and r_{max} such that all upper sequences of ranks lower than $\bar{r}_{min}$ are hypergraphic and all sequences of ranks higher than r_{max} are non-hypergraphic.

Keywords: hypergraph, degree sequence, complement

1. Introduction

A hypergraph H is a pair (V, E), where V is the vertex set of H, and E, the set of hyperedges, is a collection of non-empty subsets of V. The degree of a vertex v of H, denoted by $d(v)$, is the number of hyperedges in H containing v. A hypergraph H is *simple* if it has no repeated hyperedges. A hypergraph H is r-*uniform* if all hyperedges contain r-vertices.

Let $V = \{v_1, \cdots, v_n\}$. $d(H) = (d(v_1), \cdots, d(v_n))$ is the *degree sequence* of hypergraph H. A sequence $d = (d_1, \cdots, d_n)$ is *hypergraphic* if there is a simple hypergraph H with degree sequence d. For a given m, $0 < m \leq 2^n$, let $H_m(n)$ denote the set of all simple hypergraphs $([n], E)$, where $[n] = \{1,2, \cdots, n\}$, and

$|E| = m$. Let $D_m(n)$ denote the set of all hypergraphic sequences of hypergraphs in $H_m(n)$. The subject of our investigation is the set $D_m(n)$, as well as its complement, the set of integer n-tuples which are not hypergraphic sequences for $H_m(n)$.

The problem of characterization of $D_m(n)$ remains open even for 3-uniform hypergraphs (see [4]-[13]). The problem has its interpretation in terms of multidimensional binary cubes that arises out of the discrete isoperimetric problem for n-dimensional binary cube [1-3]. In [6] the polytope of degree sequences of uniform hypergraphs was studied and several partial results were obtained. It was shown in [10] that any two 3-uniform hypergraphs can be transformed into each other by using a sequence of trades. Several necessary and one sufficient conditions were obtained for existence of simple 3-uniform hypergraphs in [8]. Steepest degree sequences were defined in [7] and it was shown that the whole set of degree sequences of simple uniform hypergraphs can be determined by its steepest elements. Upper and lower degree sequences were defined for $D_m(n)$ in [11] where it was proven that the whole set $D_m(n)$ can be easily determined by the set of its upper and/or lower degree sequences. Upper degree sequences of the lowest and highest ranks were characterized in our earlier study [12]. In the current paper we extend the study to the complementary area of $D_m(n)$, which can be supportive in solving the problem algorithmically.

Define the grid Ξ_{m+1}^n as: $\Xi_{m+1}^n = \{(a_1, \cdots, a_n) | 0 \le a_i \le m \text{ for all } i\}$, and place a component-wise partial order on Ξ_{m+1}^n: $(a_1, \cdots, a_n) \le (b_1, \cdots, b_n)$ if and only if $a_i \le b_i$ for all i. The *rank* $r(a_1, \cdots, a_n)$ of an element $(a_1, \cdots, a_n)$ is defined as $(a_1 + \cdots + a_n)$. The Hasse diagram of Ξ_{m+1}^n has $m \cdot n + 1$ levels according to the ranks of elements: the i-th level contains all elements of the rank i. $r(a_1, \cdots, a_n) = r(b_1, \cdots, b_n) + 1$ if $(a_1, \cdots, a_n)$ covers $(b_1, \cdots, b_n)$ (see [14] for undefined terms). In this manner, $D_m(n)$ is a subset of Ξ_{m+1}^n.

A hypergraphic sequence $d = (d_1, \cdots, d_n) \in D_m(n)$, is called *upper hypergraphic* if $d_i \ge m_{mid}$ for all i, where $m_{mid} = (m+1)/2$ for odd m and $m_{mid} = m/2$ for even m. Let $\widehat{D}_m(n)$ denote the set of all upper hypergraphic sequences in $D_m(n)$. According to [11], for constructing all elements of $D_m(n)$, it is sufficient to find elements of $\widehat{D}_m(n)$, reducing in this manner the problem of describing the set of degree sequences from Ξ_{m+1}^n to $\widehat{H}$, where $\widehat{H} = \{(a_1, \cdots, a_n) | m_{mid} \le a_i \le m \text{ for all } i\}$.

Thus $\widehat{D}_m(n) = D_m(n) \cap \widehat{H}$. Figure 1 illustrates Hasse diagram of Ξ_5^3.

Figure 2 demonstrates $\widehat{D}_m(n)$ and $D_m(n)$ in Ξ_5^3.

$\widehat{F}_m(n) = \widehat{H} \backslash \widehat{D}_m(n)$ is the set of all *upper non-hypergraphic* sequences. Recall ([11]) that $\widehat{D}_m(n)$ is an ideal in $\widehat{H}$ ($\widehat{F}_m(n)$ is a filter in $\widehat{H}$).

Figure 3 illustrates $\widehat{D}_m(n)$ and $\widehat{F}_m(n)$ in $\widehat{H}$.

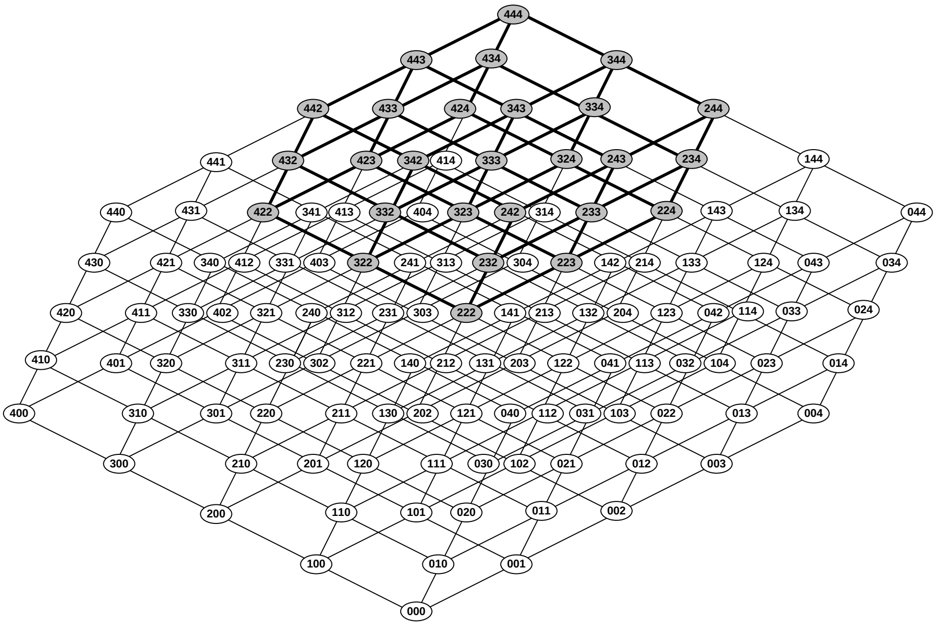

Figure 1.
Circles correspond to elements/vertices of Ξ_5^3. Highlighted part (gray) composes $\widehat{H}$.

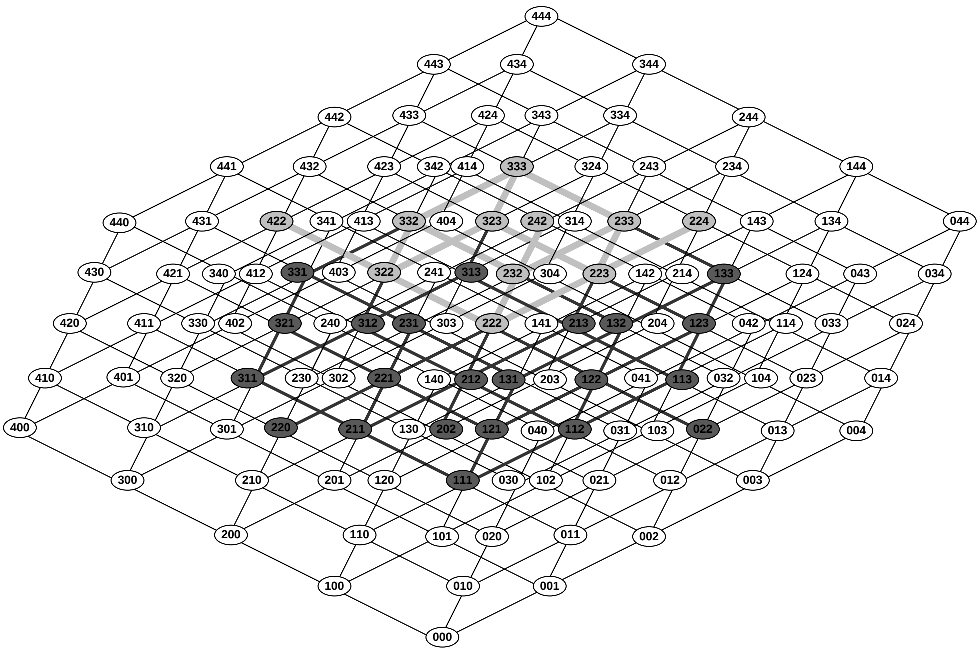

Figure 2.
Whole highlighted part composes $D_m(n)$, and its lighter part is $\widehat{D}_m(n)$.

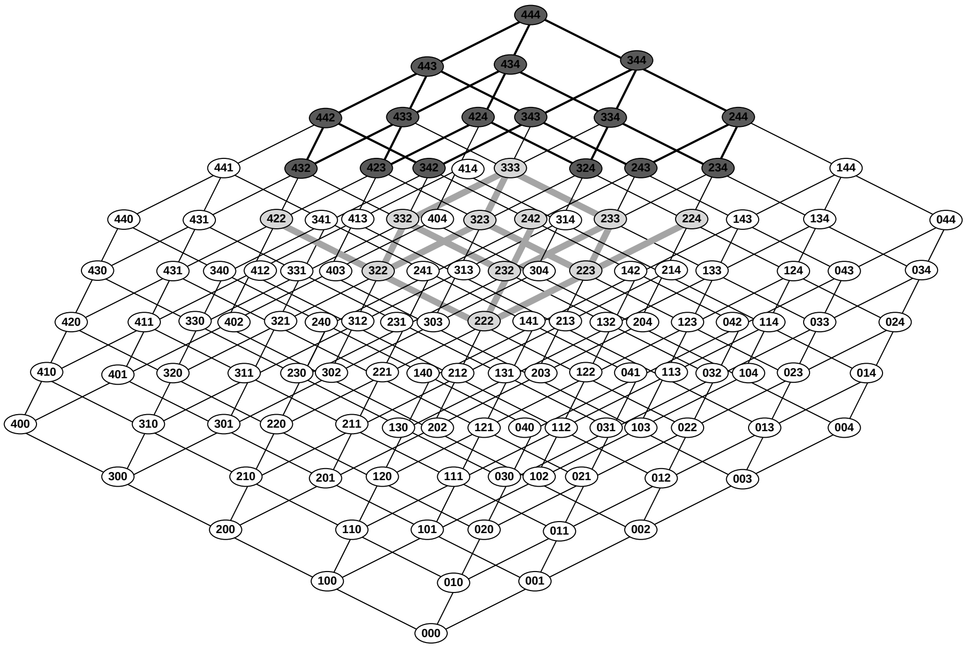

Figure 3.
Highlighted part composes $\widehat{H}$. Light part in $\widehat{H}$ is $\widehat{D}_m(n)$, and dark part is $\widehat{F}_m(n)$.

In [12] we obtained simple formulas for the lowest r_{min} and the highest r_{max} ranks of maximal elements in $\widehat{D}_m(n)$.
In this paper we present analogous results for non-hypergraphic sequences, namely we seek for the lowest and highest ranks $\bar{r}_{min}$ and $\bar{r}_{max}$, respectively, of minimal elements of $\widehat{F}_m(n)$. Section 2 determines a characterization of the lowest rank. We obtain a series of minimal elements of $\widehat{F}_m(n)$ and prove that these elements are the lowest rank non-hypergraphic sequences. Section 3 determines the highest rank minimal elements. We conclude that all sequences in $\widehat{H}$ of ranks lower than $\bar{r}_{min}$ are hypergraphic and all sequences in $\widehat{H}$ of ranks higher than r_{max} are non-hypergraphic. In the last section we give some estimates on lowest and highest ranks in $\widehat{H}$ depending on the values of m.

2. Lowest rank

In this section we provide a characterization of the lowest rank minimal elements of $\widehat{F}_m(n)$. We obtain a series of minimal elements of $\widehat{F}_m(n)$ and prove that these elements are the lowest rank non-hypergraphic sequences.
Let m be given in the standard binary representation form:

$$m = 2^{k_1} + \cdots + 2^{k_p} \text{ where } k_1 > \cdots > k_p > 0. \quad (1)$$

The lowest rank r_{min} of maximal elements of $\widehat{D}_m(n)$ is defined in [12] as follows:
$r_{min} = \sum_{i=1}^{p}((n - k_i - (i-1)) \cdot 2^{k_i} + k_i \cdot 2^{k_i - 1})$.

The maximal element d_{min} of rank r_{min} in $\widehat{D}_m(n)$ corresponds to a hypergraph whose edges are identified with the initial m-segment of the reverse lexicographic ordering of $[2^n]$ and is unique (up to coordinate permutations).

$d_i = \left(\sum_{l=1}^{j-1} 2^{k_l-1}\right) + 2^{k_j}$ for $i = k_j + 1,\ j = 1, \cdots, p,$
$d_i = \left(\sum_{l=1}^{j} 2^{k_l-1}\right) + (\sum_{l=j+1}^{p} 2^{k_j})$ for $k_{j+1} + 2 \leq i \leq k_j, j = 1, \cdots, p-1,$
$d_i = \left(\sum_{l=1}^{p} 2^{k_l-1}\right) = m/2$ for $1 \leq i \leq k_p,$
$d_i = \left(\sum_{l=1}^{p} 2^{k_l}\right) = m$ for $k_1 + 2 \leq i \leq n.$

Thus d_{min} has the following form:

$$d_{min} = \left(\overbrace{m, \cdots, m}^{n_1}, \overbrace{d_{n_1+1}, \cdots, d_{n_1+n_2}}^{n_2}, \overbrace{m_{mid}, \cdots, m_{mid}}^{n_3}\right) \tag{2}$$

where $m > d_{n_1+1} \geq \cdots \geq d_{n_1+n_2} > m_{mid}$; $n_1 + n_2 + n_3 = n$. Notice that $d_{n_1+1} = 2^{k_1}$.
Below, we establish two easily verified properties of d_{min} which will be used to prove our results.
Property 1. d_i is the largest possible value for fixed $d_1, \cdots, d_{i-1}$.
Property 2.
a) $n_1 > 0$ if and only if $m \leq 2^{n-1}$. If $2^{t-1} < m \leq 2^t$ for some $t \leq n$ then $n_1 = n - t$.
b) $n_2 = 0$ if and only if $m = 2^t$ for some t.
c) $n_3 = k_p$.

We will also use the notions of flatter and steeper elements defined as follows:

Let $a_i \geq a_j + 2$ for some $1 \leq i, j \leq n$, then $(a_1, \cdots, a_i - 1, \cdots, a_j + 1, \cdots, a_n)$ is *flatter* than $(a_1, \cdots, a_n)$ and $(a_1, \cdots, a_n)$ is *steeper* than $(a_1, \cdots, a_i - 1, \cdots, a_j + 1, \cdots, a_n)$. If $(a_1, \cdots, a_n) \in D_m(n)$, then all elements flatter than $(a_1, \cdots, a_n)$ also belong to $D_m(n)$ (see [7]).
The following theorem determines a minimal element of the lowest rank of $\hat{F}_m(n)$.

Theorem 1. Let d_{min} be presented as in (2).

(1) If $m \neq 2^t$ for arbitrary t, then:
a) $\bar{d}_{min} = \left(\overbrace{m, \cdots, m}^{n_1}, 2^{k_1} + 1, \overbrace{m_{mid}, \cdots, m_{mid}}^{n-n_1-1}\right)$ is a minimal element of $\hat{F}_m(n)$, where 2^{k_1} is the first component in (1).
b) $r(\bar{d}_{min}) = \bar{r}_{min}$.
(2) If $m = 2^t$ for some t, then:

a) $\bar{d}_{min} = \left(\overbrace{m, \cdots, m}^{n_1}, m_{mid} + 1, \overbrace{m_{mid}, \cdots, m_{mid}}^{n-n_1-1}\right)$ is a minimal element of $\hat{F}_m(n)$.
b) $r(\bar{d}_{min}) = \bar{r}_{min}$.

Proof. We consider both cases separately.
(1) $m \neq 2^t$. Then $n_2 > 0$ by Property 2.
a) Here it suffices to show that $\bar{d}_{min}$ is a non-hypergraphic sequence in $\hat{H}$ and all elements of $\hat{H}$ covered by $\bar{d}_{min}$ are hypergraphic in $\hat{H}$. $\bar{d}_{min} \notin \hat{D}_m(n)$ by Property 1. All elements of $\hat{H}$ covered by $\bar{d}_{min}$ have one of the following forms:

$$\left(\overbrace{m, \cdots, m}^{n_1}, 2^{k_1}, \overbrace{m_{mid}, \cdots, m_{mid}}^{n-n_1-1}\right) \quad (3)$$

$$\left(\overbrace{m, \cdots, m, m-1}^{n_1}, 2^{k_1} + 1, \overbrace{m_{mid}, \cdots, m_{mid}}^{n-n_1-1}\right) \quad (4)$$

Sequence (3) is less than d_{min} and hence belongs to $\hat{D}_m(n)$. If $m = 2^{k_1} + 1$, then sequence (4) is just a permutation of (3). If $m \geq 2^{k_1} + 2$, then (4) is flatter than (3), and therefore is hypergraphic.
b) We have to prove that all $a \in \hat{H}$ of $r(a) < r(\bar{d}_{min})$ belong to $\hat{D}_m(n)$. It is sufficient to show that all $a \in \hat{H}$ of $r(a) = r(\bar{d}_{min}) - 1$ belong to $\hat{D}_m(n)$. Consider $d' = \left(\overbrace{m, \cdots, m}^{n_1}, 2^{k_1}, \overbrace{m_{mid}, \cdots, m_{mid}}^{n-n_1-1}\right)$ which is of rank $r(\bar{d}_{min}) - 1$. All elements in $\hat{H}$ of the rank $r(\bar{d}_{min}) - 1$ can be obtained from d' by combinations of the following unit operations:
i) replace $(m, 2^{k_1})$ by $(m - 1, 2^{k_1} + 1)$;
ii) replace (m, m_{mid}) by $(m - 1, m_{mid} + 1)$;
iii) replace $(2^{k_1}, m_{mid})$ by $(2^{k_1} - 1, m_{mid} + 1)$.
In all three cases above the resulting sequence is either flatter than d' or is a permutation of d', and therefore belongs to $\hat{D}_m(n)$.
(2) $m = 2^t$. Then $n_2 = 0$ by Property 2.
a) Here it is only required to prove that $\bar{d}_{min} \notin \hat{D}_m(n)$ and all elements of $\hat{H}$ covered by $\bar{d}_{min}$ are hypergraphic sequences in $\hat{H}$. $\bar{d}_{min} \notin \hat{D}_m(n)$ because $\bar{d}_{min} > d_{min}$ and d_{min} is a maximal element of $\hat{D}_m(n)$.

All elements of $\hat{H}$ covered by $\bar{d}_{min}$ have one of the following forms:
$\left(\overbrace{m, \cdots, m}^{n_1}, \overbrace{m_{mid}, \cdots, m_{mid}}^{n_3}\right)$ or

$\left(\overbrace{m, \cdots, m, m-1}^{n_1}, m_{mid} + 1, \overbrace{m_{mid}, \cdots, m_{mid}}^{n_3-1}\right)$.

In the former form this is just d_{min} and in the latter form this is flatter than d_{min}, and hence belongs to $\widehat{D}_m(n)$.

b) We have to prove that all elements a of $r(a) = \bar{d}_{min} - 1$ in $\widehat{H}$ belong to $\widehat{D}_m(n)$. These are flatter than $d_{min} = \left(\overbrace{m, \cdots, m}^{n_1}, \overbrace{m_{mid}, \cdots, m_{mid}}^{n_3}\right)$, and hence belong to $\widehat{D}_m(n)$. □

Remark that in case of hypergraphic sequences there is a unique maximal element of rank r_{min} in $\widehat{D}_m(n)$, whereas in case of non-hypergraphic sequences there are a number of minimal elements of rank $\bar{r}_{min}$ in $\widehat{F}_m(n)$.
Theorem 2 below produces a series of minimal elements of $\widehat{F}_m(n)$.

Theorem 2.
(1) If $m \neq 2^t$ then

$$\left(\overbrace{m, \cdots, m, m-t}^{n_1}, 2^{k_1} + t + 1, \overbrace{m_{mid}, \cdots, m_{mid}}^{n-n_1-1}\right), \quad t = 1, \cdots, (m - 2^{k_1} - 1)/2,$$

are minimal elements of $\bar{r}_{min}$ in $\widehat{F}_m(n)$.

(2) If $m = 2^t$ then

$$\left(\overbrace{m, \cdots, m, m-t}^{n_1}, m_{mid} + t + 1, \overbrace{m_{mid}, \cdots, m_{mid}}^{n-n_1-1}\right) \; t = 1, \cdots, (m - m_{mid} - 1)/2,$$

are minimal elements of $\bar{r}_{min}$ in $\widehat{F}_m(n)$.

The proof is obtained by an analogous reasoning as in Theorem 1.

3. Highest rank

In this section we determine a characterization of highest rank minimal elements of $\widehat{F}_m(n)$.
Let m be given in the following canonical representation form:

$$m = C_n^n + C_n^{n-1} + \cdots + C_n^{n-k} + m_1, m_1 < C_n^{n-k-1} \tag{5}$$

Let the highest rank r_{max} of maximal elements of $\widehat{D}_m(n)$, be defined as in [12]:

$$r_{max} = \sum_{i=0}^{k}((n-i) \cdot C_n^{n-i}) + (n-k-1) \cdot m_1.$$

Let D_{max} denote the class of maximal elements of $\widehat{D}_m(n)$ of the rank r_{max}. D_{max} defines the set of degree sequences of that class of hypergraphs which have $C_n^n + C_n^{n-1} + \cdots + C_n^{n-k}$ common hyperedges (the subsets of $[n]$ of cardinalities $n, n-1, \cdots, n-k$) and differ only in the remaining m_1 hyperedges (the $(n-k-1)$-subsets of $[n]$). Thus, $|D_{max}| = C_{C_n^{n-k-1}}^{m_1}$. The components of all $d_{max} \in D_{max}$

are calculated as follows: $d_i = \sum_{i=0}^{k} C_{n-1}^{n-i-1} + s_i$, where $(s_1, \cdots, s_n)$ defines the set of hypergraphic sequences for $(n-k-1)$-uniform hypergraphs with m_1 edges.

Theorem 3.
Let $m = C_n^n + C_n^{n-1} + \cdots + C_n^{n-k} + m_1, m_1 < C_n^{n-k-1}$.

a) If $m_1 \geq 1 + \frac{k}{n} - k - 1$, then $\bar{r}_{max} = r_{max} + 1$.

b) If $m_1 < 1 + \frac{k}{n} - k - 1$, then $\bar{r}_{max} \leq r_{max}$.

Proof.
a) It is easy to check that there is a sequence $d' = (d_1, \cdots, d_n)$ in D_{max} such that $d_1 \geq \cdots \geq d_{n-1} > d_n$ (we can always choose m_1 hyperedges such that $s_1 \geq \cdots \geq s_{n-1} > s_n$). Consider $\bar{d}' = (d_1, \cdots, d_{n-1}, d_n + 1)$, which belongs to $\hat{F}_m(n)$. All elements covered by $\bar{d}'$ are either flatter than d' or permutations of d' and, thus, belong to $\hat{D}_m(n)$. Therefore $\bar{d}'$ is a minimal element of the rank $r_{max} + 1$ in $\hat{F}_m(n)$. Since all elements of the rank $r_{max} + 1$ in $\hat{H}$ belong to $\hat{F}_m(n)$, there is no minimal element of $\hat{F}_m(n)$ of rank higher than $r_{max} + 1$. Thus $\bar{r}_{max} = r_{max} + 1$.
b) This case is obvious.

4. Concluding remarks

The last section gives estimates on the lowest and highest ranks in $\hat{H}$ depending on parameter m and brings several concluding remarks.
As it was stated above all sequences in m with ranks lower than $\bar{r}_{min}$ are hypergraphic and all sequences in $\hat{H}$ with ranks higher than r_{max} are non-hypergraphic. Hence all maximal elements of $\hat{D}_m(n)$ and all minimal elements of $\hat{F}_m(n)$ have ranks ranging between $\bar{r}_{min}$ and $r_{max} + 1$, and, thus, are located between $\bar{r}_{min}$ and $r_{max} + 1$ levels of $\hat{H}$. An illustration of the upper hypergraphic and non-hypergraphic sequences in $\hat{H}$ for $n = 3$ and $m = 4$ is given in Figure 3.

$\hat{D}_m(n) = \{(3,3,3),$
$(4,2,2), (3,3,2), (3,2,3), (2,4,2), (2,3,3), (2,2,4),$
$(3,2,2), (2,3,2), (2,2,3),$
$(2,2,2)\}.$
Maximal elements of $\hat{D}_m(n)$ are: (3,3,3), $(4,2,2), (2,4,2), (2,2,4)$.
$\hat{F}_m(n) = \{(4,4,4),$
$(4,4,3), (4,3,4), (3,4,4),$
$(4,4,2), (4,3,3), (4,2,4), (3,4,3), (3,3,4), (2,4,4),$
$(4,3,2), (4,2,3), (3,4,2), (3,2,4), (2,4,3), (2,3,4)\}.$
Maximal elements of $\hat{F}_m(n)$ are: $(4,3,2), (4,2,3), (3,4,2), (3,2,4), (2,4,3), (2,3,4)$.
And thus $r_{min} = 8$, $r_{max} = 9$, $\bar{r}_{min} = 9$ and $\bar{r}_{max} = 9$.

We consider the lowest, highest and middle levels in $\widehat{H}$: the lowest level consists of the lowest element $(m_{mid}, \cdots, m_{mid})$, the highest level consists of the highest element $(m, \cdots, m)$ and the middle level consists of all elements of the rank $n \cdot (m + m_{mid})/2$, particularly it contains the element $((m + m_{mid})/2, \cdots, (m + m_{mid})/2)$.
Next we shall examine the distance of $\bar{r}_{min}$ and r_{max} ranks/levels from the lowest, middle and highest levels of $\widehat{H}$.
First distinguish the following cases for $\bar{r}_{min}$:

a) If $2^{t-1} < m < 2^t$ for some $t \leq n$, then:

$$\bar{d}_{min} = \left(\overbrace{m, \cdots, m}^{m-t}, 2^{t-1} + 1, \overbrace{m_{mid}, \cdots, m_{mid}}^{t-1}\right)$$
$$\bar{r}_{min} = (n - t) \cdot m + 2^{t-1} + (t - 1) \cdot m_{mid} + 1$$

The distance from the lowest level in $\widehat{H}$ will be:

$$\bar{r}_{min} - r(m_{mid}, \cdots, m_{mid}) \approx (n - t) \cdot \frac{m}{2} + \left(2^{t-1} - \frac{m}{2} + 1\right) = \frac{m}{2} \cdot (n - t - 1) + 2^{t-1} + 1$$

The distance from the middle level in $\widehat{H}$ will be:
$\bar{r}_{min} - r((m + m_{mid})/2, \cdots, (m + m_{mid})/2) \approx (n - t) \cdot \frac{m}{4} - (t - 1) \cdot \frac{m}{4} + 2^{t-1} + 1 - 3m/4 = \frac{m}{4} \cdot (n - 2t - 2) + 2^{t-1} + 1.$

b) If $m = 2^t$ for some t, then:

$$\bar{d}_{min} = \left(\overbrace{m, \cdots, m}^{m-t}, m_{mid} + 1, \overbrace{m_{mid}, \cdots, m_{mid}}^{t-1}\right)$$
$$\bar{r}_{min} = (n - t) \cdot m + t \cdot m_{mid} + 1$$

The distance from the lowest level will be:

$$\bar{r}_{min} - r(m_{mid}, \cdots, m_{mid}) \approx \frac{m}{2} \cdot (n - t) + 1$$

The distance from the middle level will be:

$\bar{r}_{min} - r((m + m_{mid})/2, \cdots, (m + m_{mid})/2) \approx (n - t) \cdot \frac{m}{4} - t \cdot \frac{m}{4} + 1 = \frac{m}{4} \cdot (n - 2t) + 1.$

As we see in both cases $\bar{r}_{min}$ goes up from the lowest level in $\widehat{H}$ with decrease of m.

Now we estimate the case of r_{max}. We have $r_{max} = n \cdot \sum_{i=0}^{k} C_{n-1}^{n-i-1} + m_1 \cdot (n-k-1)$.

a) $m_1 = 0$. $m = \sum_{i=0}^{k} C_n^{n-i} = 2 \cdot \sum_{i=0}^{k-1} C_{n-1}^{n-i-1} + C_{n-1}^{n-k-1}$, and thus $m_{mid} = \sum_{i=0}^{k-1} C_{n-1}^{n-i-1} + C_{n-1}^{n-k-1}/2$.

On the other hand, all components of d_{max} are:

$d_i = \sum_{i=0}^{k} C_{n-1}^{n-i-1}$.

The distance from the lowest level in $\widehat{H}$ element is:

$r_{max} - r(m_{mid}, \cdots, m_{mid}) = C_{n-1}^{n-k-1} \cdot n/2$.

The distance from the highest level is:

$r(m, \cdots, m) - r_{max} = n \cdot \left(2 \cdot \sum_{i=0}^{k-1} C_{n-1}^{n-i-1} + C_{n-1}^{n-k-1}\right) - n \cdot \sum_{i=0}^{k} C_{n-1}^{n-i-1} = n \cdot \sum_{i=0}^{k-1} C_{n-1}^{n-i-1}$.

b) $m_1 > 0$. $m = \sum_{i=0}^{k} C_n^{n-i} + m_1 = 2 \cdot \sum_{i=0}^{k-1} C_{n-1}^{n-i-1} + C_{n-1}^{n-k-1} + m_1$, and thus $m_{mid} = \sum_{i=0}^{k-1} C_{n-1}^{n-i-1} + (C_{n-1}^{n-k-1} + m_1)/2$.

The distance from the lowest level is:

$r_{max} - r(m_{mid}, \cdots, m_{mid}) = C_{n-1}^{n-k-1} \cdot \frac{n}{2} + m_1 \cdot\cdot (n-k-1) - n \cdot \frac{m_1}{2} = C_{n-1}^{n-k-1} \cdot \frac{n}{2} + m_1(\frac{n}{2} - k - 1)$.

The distance from the highest level is:

$n \cdot \sum_{i=0}^{k-1} C_{n-1}^{n-i-1} - m_1 \cdot (n-k-1)$.

Thus we have determined the layouts of r_{max} and $\bar{r}_{min}$ in $\widehat{H}$ depending on parameter m. The obtained formulas show when r_{max} and $\bar{r}_{min}$ are close, or when $\bar{r}_{min}$ is greater than the middle rank depending on m, etc.

References

[1] L. A. Aslanyan, Isoperimetry problem and related extremal problems of discrete spaces, Problemy Kibernet. 36 (1979) 85-126.

[2] L. Aslanyan, H. Danoyan, Complexity of Hash-Coding type search algorithms with perfect codes, Journal of Next Generation Information Technology (JNIT), Volume 5, Number 4, November 2014, 26-35.

[3] L. Aslanyan, H. Danoyan, On the optimality of the Hash-Coding type nearest neighbour search algorithm", Selected Revised Papers of 9th CSIT conference, submited to IEEE Xplore, 2013.
http://dx.doi.org/10.1109/csitechnol.2013.6710336

[4] L. Aslanyan, H. Sahakyan, Numerical characterization of n-cube subset partioning, Electronic Notes in Discrete Mathematics, Volume 27, Pages 3-4/10, October 2006, Elsevier B.V.,ODSA 2006 -Conference on Optimal Discrete Structures and algorithms. http://dx.doi.org/10.1016/j.endm.2006.08.027

[5] C. Berge, Hypergraphs. North Holland, Amsterdam, 1989.

[6] N. L. Bhanu Murthy and Murali K. Srinivasan, The polytope of degree sequences of hypergraphs. Linear algebra and its applications, 350 (2002), 147-170. http://dx.doi.org/10.1016/s0024-3795(02)00272-0

[7] D. Billington, Lattices and Degree Sequences of Uniform Hypergraphs. Ars Combinatoria, 21A, 1986, 9-19.

[8] D. Billington, Conditions for degree sequences to be realisable by 3-uniform hypergraphs, 1988.

[9] C. J. Colbourn, W. L. Kocay, and D. R. Stinson, Some NP-complete problems for hypergraph degree sequences, Discrete Applied Maths. 14 (1986), 239-254. http://dx.doi.org/10.1016/0166-218x(86)90028-4

[10] W. Kocay, P. C. Li, On 3-hypergraphs with equal degree sequences, Ars Combin. 82 (2007), 145-157.

[11] H. Sahakyan, Numerical characterization of n-cube subset partitioning, Discrete Applied Mathematics 157 (2009) 2191-2197. http://dx.doi.org/10.1016/j.dam.2008.11.003

[12] H. Sahakyan, Essential points of the n-cube subset partitioning characterization, Discrete Applied Mathematics 163, Part 2 (2014) 205-213. http://dx.doi.org/10.1016/j.dam.2013.07.015

[13] H. Sahakyan, (0, 1)-matrices with different rows, Selected Revised Papers of 9th CSIT conference, submited to IEEE Xplore, 2013. http://dx.doi.org/10.1109/csitechnol.2013.6710342

[14] R. Steven, Lattices and Ordered Sets, 2008, ISBN 0-387-78900-2, 305. http://dx.doi.org/10.1007/978-0-387-78901-9

Received: December 1, 2014; Published: January 3, 2015